

Weyl functions of bounded quasi-selfadjoint operators and block operator Jacobi matrices

YURY ARLINSKIĬ and LUTZ KLOTZ

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Abstract. A bounded operator T in a separable Hilbert space $\mathfrak{H}$ is called quasi-selfadjoint if $\ker(T - T^*) \neq \{0\}$ and $\mathfrak{N}$ -quasi-selfadjoint if $\mathfrak{N} \supseteq \operatorname{ran}(T - T^*)$, where $\mathfrak{N}$ is a subspace of $\mathfrak{H}$. An $\mathfrak{N}$ -quasi-selfadjoint operator T is called $\mathfrak{N}$ -simple if the linear hull of $\{T^n \mathfrak{N}, n = 0, 1, \dots\}$ is dense in $\mathfrak{H}$. We study the $\mathfrak{N}$ -Weyl function $M(z) = P_{\mathfrak{N}}(T - zI_{\mathfrak{H}})^{-1} \upharpoonright \mathfrak{N}$ of an $\mathfrak{N}$ -quasi-selfadjoint operator and define its so-called “Schur parameters”. The main result of the paper is that any $\mathfrak{N}$ -quasi-selfadjoint and $\mathfrak{N}$ -simple operator is unitarily equivalent to an operator given by a special block operator Jacobi matrix constructed by means of the Schur parameters of its $\mathfrak{N}$ -Weyl function.

1. Notations

We use the common symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}_0$ for the sets of complex numbers, real numbers, positive integers, and nonnegative integers, respectively. For a complex number z , the notation $\bar{z}$ means its complex conjugate.

The closure of a subset M of a topological space is denoted by $\overline{M}$. If M is a subset of a topological vector space, then $\operatorname{span} M$ denotes its linear hull and $\overline{\operatorname{span} M}$ is the closure of $\operatorname{span} M$. All Hilbert spaces we shall consider in this paper are separable spaces over $\mathbb{C}$. The inner product and corresponding norm of a Hilbert space $\mathfrak{H}$ are denoted by $(\cdot, \cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$, respectively. We omit the symbol $\mathfrak{H}$ in these notations if there is no danger of confusion.

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Throughout this paper the notion “operator” means “bounded linear operator”. The linear space of operators acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is denoted by $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and the Banach algebra $\mathbf{L}(\mathfrak{H}, \mathfrak{H})$ by $\mathbf{L}(\mathfrak{H})$. The domain, the range, and the null-space of an operator T are denoted by $\text{dom } T$, $\text{ran } T$, and $\text{Ker } T$, respectively, and $\overline{\text{ran } T}$ stands for the closure of $\text{ran } T$. If $T \in \mathbf{L}(\mathfrak{H}, \mathfrak{K})$, then T^* and $\|T\|$ denote its adjoint and norm, respectively. For any subset M of $\mathfrak{H}$, define $TM := \{Tf, f \in M\}$. If $\mathcal{L}$ is a subspace, i.e. a closed linear subset, of $\mathfrak{H}$, the orthogonal projection in $\mathfrak{H}$ onto $\mathcal{L}$ is denoted by $P_{\mathcal{L}}$ and the restriction of T to $\mathcal{L}$ by $T|_{\mathcal{L}}$. If T is invertible, then T^{-1} denotes its inverse. If $T \in \mathbf{L}(\mathfrak{H})$, the operators $T_R = (T + T^*)/2$ and $T_I = (T - T^*)/2i$ are said to be its *real* and *imaginary parts*, respectively. By $\rho(T)$ and $\sigma(T)$ we denote the set of its regular points and spectrum, respectively. The identity operator in a Hilbert space $\mathfrak{H}$ is denoted by $I_{\mathfrak{H}}$. The symbol $R_T(z) = (T - zI_{\mathfrak{H}})^{-1}$, $z \in \rho(T)$, stands for the resolvent of T . A selfadjoint operator T in a Hilbert space $\mathfrak{H}$ is called nonnegative if $(Tf, f) \geq 0$ for all $f \in \mathfrak{H}$. In this case we shall often use the notation $T \geq 0$. If $(Tf, f) > 0$ for all $f \in \mathfrak{H} \setminus \{0\}$, we shall write $T > 0$. A nonnegative operator T is called positive definite if $0 \in \rho(T)$.

2. Introduction

In order to describe the main objectives and results of the present paper we first recall the notions of *Schur transformation* [35] and *Schur parameters* for functions analytic on the open unit disk and bounded by one (the Schur class $\mathbf{S}$).

Given $f \in \mathbf{S}$, define inductively

$$f_0(\lambda) = f(\lambda), \quad f_{n+1}(\lambda) = \frac{f_n(\lambda) - f_n(0)}{\lambda(1 - \overline{f_n(0)}f_n(\lambda))}, \quad n \geq 0, \quad |\lambda| < 1.$$

It is clear that $\{f_n\}$ is a finite or infinite sequence of functions from the class $\mathbf{S}$. We recall that it is a finite sequence if and only if f can be written as a finite Blaschke product. The numbers $\gamma_n := f_n(0)$ are called the *Schur parameters*. The method of labeling $f \in \mathbf{S}$ by its Schur parameters is known as the *Schur algorithm* and is due to I. Schur [35]. It is proved in [35] that the Schur parameters uniquely determine the Schur function. The transformation

$$\mathbf{S} \ni f \mapsto \varphi \in \mathbf{S} : \quad \varphi(\lambda) = \frac{f(\lambda) - f(0)}{\lambda(1 - \overline{f(0)}f(\lambda))}$$

is now called the *Schur transformation* [5]. The Schur algorithm for matrix valued Schur class functions has been considered in the paper of Delsarte, Genin, and

Kamp [27] and in the book of Dubovoj, Fritzsche, and Kirstein [28]. An operator extension of the Schur algorithm was developed by T. Constantinescu in [23] and is presented with numerous applications in the book of Bakonyi and Constantinescu [16].

Recall that a function g belongs to the Herglotz–Nevanlinna class if it is holomorphic on the open upper half plane and

$$\frac{\operatorname{Im} g(z)}{\operatorname{Im} z} \geq 0.$$

Setting $g(z) = \overline{g(\bar{z})}$, $\operatorname{Im} z < 0$, one can extend g to a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$, which is also denoted by g and called Herglotz–Nevanlinna function. Let g be a Herglotz–Nevanlinna function such that

$$g(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} + o\left(\frac{1}{z^2}\right), \quad z = iy, \quad y \rightarrow \pm\infty.$$

In [1, Lemma 3.3.6] it is proved that if $s_0 = 0$, then $g = 0$ and if $s_0 > 0$, then the function g_1 :

$$g_1(z) = \frac{s_1}{s_0} - z - \frac{s_0}{g(z)}$$

is a Herglotz–Nevanlinna function such that $\lim_{y \rightarrow \infty} g_1(iy) = 0$. The transformation

$$g \mapsto g_1$$

is called the *Schur transformation for Herglotz–Nevanlinna functions* [5], [6].

Let $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$. Denote by $\mathbf{N}^0[\alpha, \beta]$ the set of all Herglotz–Nevanlinna functions m which are holomorphic on $\mathbb{C} \setminus [\alpha, \beta]$ and such that

$$\lim_{z \rightarrow \infty} zm(z) = -1.$$

Any function $m \in \mathbf{N}^0[\alpha, \beta]$ has an integral representation

$$(2.1) \quad m(z) = \int_{\alpha}^{\beta} \frac{d\sigma(t)}{t - z},$$

where σ is a nondecreasing function on $[\alpha, \beta]$ such that $\sigma(\alpha) = 0$ and $\sigma(\beta) = 1$. It follows that m has an asymptotic expansion

$$(2.2) \quad m(z) = -\frac{1}{z} - \frac{b_0}{z^2} - \frac{a_0^2 + b_0^2}{z^3} + o\left(\frac{1}{z^3}\right), \quad z \rightarrow \infty,$$

and its Schur transformation m_1 :

$$(2.3) \quad m_1(z) = \frac{1}{a_0^2} \left(b_0 - z - \frac{1}{m(z)} \right)$$

belongs to the class $\mathbf{N}^0[\alpha, \beta]$ as well (see [29]). Thus, the Schur transformation preserves the class $\mathbf{N}^0[\alpha, \beta]$. A repeated application of the Schur transformation gives a representation of m by a finite or infinite continued fraction (J-fraction, cf. [38])

$$(2.4) \quad m(z) = \frac{-1}{z - b_0} + \frac{-a_0^2}{z - b_1} + \frac{-a_1^2}{z - b_2} + \dots + \frac{-a_{n-1}^2}{z - b_n} + \dots$$

The process terminates at the N -th step if and only if m is a rational function of the form

$$m(z) = \sum_{k=1}^N \frac{c_k}{\mu_k - z},$$

where $\{\mu_k\}_{k=1}^N$ are real numbers and $\{c_k\}_{k=1}^N$ are positive numbers such that $\sum_{k=1}^N c_k = 1$.

Recall that a selfadjoint operator T in a separable Hilbert space $\mathfrak{H}$ is called prime if there exists a cyclic vector $e \in \mathfrak{H}$ of T (cf. [2]). If e is a normalized cyclic vector of T , then the function

$$(2.5) \quad m(z) = ((T - zI_{\mathfrak{H}})^{-1}e, e)_{\mathfrak{H}}, \quad z \in \rho(T),$$

is called *Weyl's m -function* (or *Weyl function* for short) of T (cf. [29]). It is a well known fact that the set of Weyl functions of prime selfadjoint operators in separable Hilbert spaces coincides with the function class

$$\mathbf{N}^0 := \bigcup_{[\alpha, \beta] \subset \mathbb{R}} \mathbf{N}^0[\alpha, \beta].$$

Moreover, the Weyl function m of a selfadjoint operator T belongs to the class $\mathbf{N}^0[\alpha, \beta]$, where

$$\alpha = \inf\{(Tf, f), f \in \mathfrak{H}, \|f\| = 1\}, \quad \beta := \sup\{(Tf, f), f \in \mathfrak{H}, \|f\| = 1\}.$$

According to Stone's theorem every prime, even unbounded, selfadjoint operator T can be represented in some orthonormal basis by a finite or semi-infinite Jacobi

matrix of the form

$$(2.6) \quad J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdot & \cdot & \cdot \\ a_0 & b_1 & a_1 & 0 & \cdot & \cdot & \cdot \\ 0 & a_1 & b_2 & a_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1} & b_n \end{pmatrix}$$

or

$$(2.7) \quad J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdot & \cdot \\ a_0 & b_1 & a_1 & 0 & 0 & \cdot & \cdot \\ 0 & a_1 & b_2 & a_2 & 0 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_k > 0$, and b_k are real numbers for all $k = 0, 1, \dots$ (see [1], [2]). If T is bounded and m is its Weyl function, the entries of the corresponding Jacobi matrix are the numbers from the right hand side of (2.4). On the other hand, the same numbers appear in the three term recurrence relations

$$(2.8) \quad \begin{aligned} a_k P_{k+1}(z) + b_k P_k(z) + a_{k-1} P_{k-1}(z) &= z P_k(z), \quad k \geq 0, \quad a_1 = 1, \\ P_1(z) &= 0, \quad P_0(z) = 1 \end{aligned}$$

for polynomials $\{P_k\}$ orthonormalized with respect to the measure $d\sigma$ in (2.1).

Notice that the Schur transformation has been extended to scalar generalized Nevanlinna and Schur classes of functions by M. Derevyagin in [25], M. Derevyagin and V. Derkach in [26], and D. Alpay, T. Azizov, A. Dijksma, H. Langer, Yu. Shondin, and G. Wanjala in [4], [8], [9], [5], [6], [7], [10]. For matrix-valued Herglotz–Nevanlinna functions the Schur algorithm has been applied to the study of interpolation problems and truncated moment problems by I. Kovalishina [31], V. Bolotnikov [18], and G.-N. Chen and Y.-J. Hu [22].

In the paper [13] it is proved that every prime bounded operator T with rank one imaginary part T_I in a separable Hilbert space $\mathfrak{H}$ can be represented by a Jacobi matrix of the form (2.6) or (2.7) with respect to some orthonormal basis. The only difference with the selfadjoint case is that the number b_0 is nonreal. The method used in [13] is based on the Livšic characteristic function [33], [20] or on the Weyl function m of T given by (2.5), where $e \in \text{ran } T_I$. In this case m is no longer a Herglotz–Nevanlinna function but has an asymptotic expansion (2.2), where $a_0 > 0$ and b_0 is a nonreal number. Nevertheless, its Schur transformation (2.3) belongs to the class $\mathcal{N}^0$. Thus, m can be expanded into a continued fraction

of the form (2.4) and, moreover, the Jacobi matrix corresponding to T has entries coming from (2.4). An intrinsic characterization of the Weyl functions of bounded prime dissipative operators with rank one imaginary part has been established in [13].

The present paper deals with bounded operators T in a separable Hilbert space $\mathfrak{H}$ satisfying the condition

$$\text{Ker } T_I \neq \{0\}.$$

Such operators are called *quasi-selfadjoint*. If $\mathfrak{N}$ is a subspace of $\mathfrak{H}$ such that $\mathfrak{N} \supseteq \overline{\text{ran}} T_I$, we call the operator T an $\mathfrak{N}$ -quasi-selfadjoint operator and define its $\mathfrak{N}$ -Weyl function $M_{T,\mathfrak{N}}$ by

$$M_{T,\mathfrak{N}}(z) = P_{\mathfrak{N}} R_T(z)|_{\mathfrak{N}}, \quad z \in \rho(T).$$

The function $M_{T,\mathfrak{N}}$ is a Herglotz–Nevanlinna function if and only if T is selfadjoint. Moreover, in this case $M_{T,\mathfrak{N}}$ belongs to the subclass $\mathbf{N}_{\mathfrak{N}}^0$ of $\mathbf{L}(\mathfrak{N})$ -valued Herglotz–Nevanlinna functions holomorphic in a neighborhood of infinity and such that $\lim_{z \rightarrow \infty} z M_{T,\mathfrak{N}}(z) = -I_{\mathfrak{N}}$. The $\mathfrak{N}$ -Weyl functions of $\mathfrak{N}$ -quasi-selfadjoint operators form a larger class, which is denoted by $\mathbf{M}_{\mathfrak{N}}^0$ and defined in Definition 5.10.

Let $M_0 \in \mathbf{M}_{\mathfrak{N}}^0$. Define the operators

$$\begin{aligned} B_0 &= - \lim_{z \rightarrow \infty} z^2 (M_0(z) + I_{\mathfrak{N}} z^{-1}), \\ D_0 &= - \lim_{z \rightarrow \infty} z^3 (M_0(z) + I_{\mathfrak{N}} z^{-1} + B z^{-2}) - B_0^2. \end{aligned}$$

We shall see that D_0 is a nonnegative operator. Let $\mathfrak{N}_1 := \overline{\text{ran}} D_0$. We establish (see Theorem 6.1) that there exists a unique Herglotz–Nevanlinna function M_1 from the class $\mathbf{N}_{\mathfrak{N}_1}^0$ such that in a neighborhood of infinity one has

$$M_0^{-1}(z) = B_0 - z I_{\mathfrak{N}} - D_0^{1/2} M_1(z) D_0^{1/2}.$$

The function M_1 will be called the Schur transformation of M_0 .

Now we apply the preceding construction to the function M_1 instead of M_0 and continue this procedure. If $\mathfrak{N}_{N+1} = \{0\}$ ($\iff D_N = 0$), then the process terminates at the N -th step. Otherwise the process is infinite. As a result we obtain a finite or infinite chain of Hilbert spaces

$$\mathfrak{N} = \mathfrak{N}_0 \supseteq \mathfrak{N}_1 \supseteq \mathfrak{N}_2 \supseteq \cdots,$$

bounded selfadjoint operators $B_k \in \mathbf{L}(\mathfrak{N}_k)$, $k \geq 1$, bounded nonnegative operators $D_k \in \mathbf{L}(\mathfrak{N}_k)$, $k \geq 0$, such that $\overline{\text{ran}} D_k = \mathfrak{N}_{k+1}$, and a sequence of functions $M_k \in \mathbf{N}_{\mathfrak{N}_k}^0$, $k \geq 1$, which are defined in a common neighborhood of infinity U , satisfy the equalities

$$M_k^{-1}(z) = B_k - zI_{\mathfrak{N}_k} - D_k^{1/2}M_{k+1}(z)D_k^{1/2}, \quad z \in U,$$

and have the asymptotic behavior

$$M_k(z) = -I_{\mathfrak{N}_k}z^{-1} - B_kz^{-2} - (B_k^2 + D_k)z^{-3} + o(z^{-3}), \quad z \rightarrow \infty, \quad k \geq 0.$$

The above sequences of Hilbert spaces $\{\mathfrak{N}_k\}$ and operators $\{B_k\}$ and $\{D_k\}$ we call the sequences of *Schur parameters* of M_0 .

Define a Hilbert space $\mathfrak{H}$ by

$$\mathfrak{H} = \sum_{k \geq 0} \oplus \mathfrak{N}_k.$$

Put $A_k := D_k^{1/2} \restriction \mathfrak{N}_{k+1}$ and consider A_k as elements of $\mathbf{L}(\mathfrak{N}_{k+1}, \mathfrak{N}_k)$, $k \geq 0$. By means of the operators $\{B_k\}$ and $\{A_k\}$ we construct an operator in $\mathfrak{H}$ given by the block operator Jacobi matrix

$$J_0 = \begin{pmatrix} B_0 & A_0 & 0 & 0 & 0 & \dots \\ A_0^* & B_1 & A_1 & 0 & 0 & \dots \\ 0 & A_1^* & B_2 & A_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The operator J_0 is $\mathfrak{N}$ -quasi-selfadjoint and $\mathfrak{N}$ -simple, i.e. $\overline{\text{span}} \{T^n \mathfrak{N}, n \in \mathbb{N}_0\} = \mathfrak{H}$. We prove that the $\mathfrak{N}$ -Weyl function of J_0 coincides with M_0 in some neighborhood of infinity. We mention the following consequences of our result, which can be considered as analogs of Stone's theorem [36]:

- every bounded selfadjoint operator which is simple with respect to a subspace is unitarily equivalent to a selfadjoint block operator Jacobi matrix, in particular, if a bounded selfadjoint operator has a spectrum of multiplicity $m < \infty$ (see [2]), then it is unitarily equivalent to a selfadjoint block operator Jacobi matrix,
- every completely non-selfadjoint [20] and quasi-selfadjoint operator is unitarily equivalent to a quasi-selfadjoint block operator Jacobi matrix.

The latter result gives an alternative model to the Livšic triangular model of a bounded linear operator with nuclear imaginary part (see [33] and [21]). We establish a connection between the class $\mathbf{M}_{\mathfrak{N}}^0$ and 1) the class $\Omega_{\mathcal{J}}$ of characteristic

functions of the Livšic–Brodskiĭ operator colligations [20], 2) the class of transfer functions of discrete time-invariant quasi-selfadjoint linear systems [19], [12].

In the case of $\mathfrak{N} = \mathfrak{N}_1 = \dots = \mathbb{C}^m$ the selfadjoint bounded and unbounded block Jacobi matrices arise in the theory of matrix moment problems and in the theory of matrix orthogonal polynomials on the real line (see [32], [17], [10], [24], and references therein).

3. Weyl functions

Definition 3.1. Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{N}$ be a non-zero subspace of $\mathfrak{H}$ (the case $\mathfrak{N} = \mathfrak{H}$ is not excluded). Let $T \in \mathbf{L}(\mathfrak{H})$. The function

$$M_{T,\mathfrak{N}}(z) = P_{\mathfrak{N}}R_T(z)|_{\mathfrak{N}}, \quad z \in \rho(T),$$

is called *the $\mathfrak{N}$ -Weyl function of T* .

Clearly,

$$(3.1) \quad M_{T,\mathfrak{N}}^*(z) = M_{T^*,\mathfrak{N}}(\bar{z}), \quad z \in \rho(T).$$

Expanding $M_{T,\mathfrak{N}}$ into the Taylor series at infinity, we get

$$(3.2) \quad M_{T,\mathfrak{N}}(z) = - \sum_{j=1}^{\infty} X_j z^{-j}, \quad |z| > \|T\|,$$

where

$$X_j = P_{\mathfrak{N}}T^{j-1}|_{\mathfrak{N}}, \quad j \in \mathbb{N}.$$

The series (3.2) converges with respect to the operator norm topology. If $\mathfrak{N} = \mathfrak{H}$, then, clearly, $M_{T,\mathfrak{H}}(z) = R_T(z)$ for all $z \in \rho(T)$. If $\mathfrak{N} \neq \mathfrak{H}$, set $\mathcal{H} = \mathfrak{H} \ominus \mathfrak{N}$ and consider the operator block matrix representation

$$(3.3) \quad T = \begin{pmatrix} B & D \\ C & A \end{pmatrix} : \begin{array}{c} \mathfrak{N} \\ \mathcal{H} \end{array} \rightarrow \begin{array}{c} \mathfrak{N} \\ \mathcal{H} \end{array}.$$

Observe that the entries of this matrix are operators whose norms do not exceed the norm of T . Expressing the first three coefficients of (3.2) in terms of these entries we obtain the asymptotic expansion

$$(3.4) \quad M_{T,\mathfrak{N}}(z) = -I_{\mathfrak{N}}z^{-1} - Bz^{-2} - (B^2 + DC)z^{-3} + o(z^{-3}), \quad z \rightarrow \infty.$$

It follows that B and DC can be recovered from $M_{T,\mathfrak{N}}$ by the relations

$$(3.5) \quad B = - \lim_{z \rightarrow \infty} z^2 (M_{T,\mathfrak{N}}(z) + I_{\mathfrak{N}} z^{-1}),$$

and

$$(3.6) \quad DC = - \lim_{z \rightarrow \infty} z^3 (M_{T,\mathfrak{N}}(z) + I_{\mathfrak{N}} z^{-1} + B z^{-2}) - B^2.$$

The next result is known as the Schur–Frobenius formula, cf. [28, Lemma 1.1.7]. It is a cornerstone of the present paper.

Theorem 3.2. *Let T be given by (3.3) and let*

$$(3.7) \quad V(z) := zI_{\mathfrak{N}} - B + DR_A(z)C, \quad z \in \rho(A).$$

Then

$$z \in \rho(T) \cap \rho(A) \iff V^{-1}(z) \in \mathbf{L}(\mathfrak{N}),$$

and then

$$(3.8) \quad R_T(z) = \begin{pmatrix} -V^{-1}(z) & V^{-1}(z)DR_A(z) \\ R_A(z)CV^{-1}(z) & R_A(z)(I_{\mathcal{H}} - CV^{-1}(z)DR_A(z)) \end{pmatrix},$$

where $z \in \rho(T) \cap \rho(A)$.

Applying (3.8) we immediately obtain

$$(3.9) \quad M_{T,\mathfrak{N}}(z) = -V^{-1}(z), \quad z \in \rho(T) \cap \rho(A),$$

which in turn yields the existence of $M_{T,\mathfrak{N}}^{-1}(z)$ if $z \in \rho(T) \cap \rho(A)$.

Definition 3.3. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces and let $\mathfrak{N} \subseteq \mathfrak{H}_1 \cap \mathfrak{H}_2$ be a subspace. Two operators $T_1 \in \mathbf{L}(\mathfrak{H}_1)$ and $T_2 \in \mathbf{L}(\mathfrak{H}_2)$ are called $\mathfrak{N}$ -unitarily equivalent if there exists a unitary operator $U \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ such that

$$U \upharpoonright \mathfrak{N} = I_{\mathfrak{N}} \quad \text{and} \quad UT_1 = T_2U.$$

The following result can be immediately derived from (3.9) and (3.7).

Proposition 3.4. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces and let $\mathfrak{N} \subseteq \mathfrak{H}_1 \cap \mathfrak{H}_2$ be a subspace. If the operators $T_k \in \mathbf{L}(\mathfrak{H}_k)$, $k = 1, 2$, are $\mathfrak{N}$ -unitarily equivalent, then $W_{T_1, \mathfrak{N}} = W_{T_2, \mathfrak{N}}$.*

We conclude this section with a useful resolvent formula.

Proposition 3.5. *Let $T, S \in \mathbf{L}(\mathfrak{H})$. Then for $z \in \rho(T + S) \cap \rho(T)$, one has*

$$(I_{\mathfrak{H}} + SR_T(z))^{-1} \in \mathbf{L}(\mathfrak{H})$$

and

$$(3.10) \quad R_{T+S}(z) = R_T(z) - R_T(z)(I_{\mathfrak{H}} + SR_T(z))^{-1}SR_T(z).$$

Proof. Since $T + S - zI_{\mathfrak{H}} = (I_{\mathfrak{H}} + SR_T(z))(T - zI_{\mathfrak{H}})$, it follows that $(I_{\mathfrak{H}} + SR_T(z))^{-1} \in \mathbf{L}(\mathfrak{H})$ and $R_{T+S}(z) = R_T(z)(I_{\mathfrak{H}} + SR_T(z))^{-1}$ for $z \in \rho(T + S) \cap \rho(T)$.

On the other hand,

$$R_{T+S}(z) - R_T(z) = R_{T+S}(z)(T - zI_{\mathfrak{H}} - (T + S - zI_{\mathfrak{H}}))R_T(z) = -R_{T+S}(z)SR_T(z).$$

Hence, we obtain (3.10). ■

The following corollary of Proposition 3.5 has been established in [11, Theorem 5.1].

Corollary 3.6. *If $T \in \mathbf{L}(\mathfrak{H})$ and $S \in \mathbf{L}(\mathfrak{N}, \mathfrak{H})$, then*

$$R_{T+SP_{\mathfrak{N}}}(z) = R_T(z) - R_T(z)S(I_{\mathfrak{N}} + P_{\mathfrak{N}}R_T(z)S)^{-1}P_{\mathfrak{N}}R_T(z),$$

where $z \in \rho(T + SP_{\mathfrak{N}}) \cap \rho(T)$.

Proof. Apply Proposition 3.5 to the operators T and $SP_{\mathfrak{N}}$. Note that $(I_{\mathfrak{N}} + P_{\mathfrak{N}}R_T(z)S)^{-1} \in \mathbf{L}(\mathfrak{N})$ if and only if $(I_{\mathfrak{H}} + SP_{\mathfrak{N}}R_T(z))^{-1} \in \mathbf{L}(\mathfrak{H})$, and in this case

$$S(I_{\mathfrak{N}} + P_{\mathfrak{N}}R_T(z)S)^{-1} = (I_{\mathfrak{H}} + SP_{\mathfrak{N}}R_T(z))^{-1}S. \quad \blacksquare$$

4. Simple and quasi-selfadjoint operators

Definition 4.1. Let $\mathfrak{N}$ be a non-zero subspace of $\mathfrak{H}$ and $T \in \mathbf{L}(\mathfrak{H})$. The operator T is called $\mathfrak{N}$ -simple if

$$\overline{\text{span}} \{T^n \mathfrak{N}, n \in \mathbb{N}_0\} = \mathfrak{H}.$$

Observe that the Taylor expansion of the resolvent

$$R_T(z) = - \sum_{j=0}^{\infty} T^j z^{-j-1}, \quad |z| > \|T\|,$$

implies that

$$(4.1) \quad \overline{\text{span}} \{T^n \mathfrak{N}, n \in \mathbb{N}_0\} = \overline{\text{span}} \{R_T(z) \mathfrak{N}, |z| > r\}$$

for any $r \geq \|T\|$. Clearly, any operator $T \in \mathbf{L}(\mathfrak{H})$ is $\mathfrak{H}$ -simple.

Proposition 4.2. Let $\mathfrak{N}$ be a proper non-zero subspace of $\mathfrak{H}$. Let $T \in \mathbf{L}(\mathfrak{H})$ and (3.3) be the corresponding matrix representation of T . Then the operator T is $\mathfrak{N}$ -simple if and only if the operator A in $\mathcal{H}$ is $\overline{\text{ran}} C$ -simple.

Proof. The Schur–Frobenius formula (3.8) implies the relation

$$R_T(z) \mathfrak{N} = \left\{ \begin{pmatrix} f \\ R_A(z) C f \end{pmatrix}, f \in \mathfrak{N} \right\}, \quad |z| > \|T\|.$$

Denote $\mathfrak{H}' = \overline{\text{span}} \{T^n \mathfrak{N}, n \in \mathbb{N}_0\}$. Then $\mathfrak{N} \subset \mathfrak{H}'$ and, hence,

$$\mathfrak{H}' = \mathfrak{N} \oplus \overline{\text{span}} \{R_A(z) C \mathfrak{N}, |z| > \|T\|\},$$

which gives

$$\mathfrak{H}' = \mathfrak{N} \oplus \overline{\text{span}} \{A^n C \mathfrak{N}, n \in \mathbb{N}_0\}.$$

by (4.1). Therefore, $\mathfrak{H}' = \mathfrak{H}$ if and only if $\overline{\text{span}} \{A^n C \mathfrak{N}, n \in \mathbb{N}_0\} = \mathcal{H}$. ■

The preceding proposition implies that $\mathfrak{N}$ -simplicity of T does not depend on the operator $P_{\mathfrak{N}} T$.

Definition 4.3. An operator $T \in \mathbf{L}(\mathfrak{H})$ is called *quasi-selfadjoint* if

$$\text{Ker } T_I \neq \{0\}.$$

Definition 4.4. Let $\mathfrak{N} \subseteq \mathfrak{H}$ and let $T \in \mathbf{L}(\mathfrak{H})$. The operator T is called $\mathfrak{N}$ -quasi-selfadjoint if

$$\text{ran } T_I \subseteq \mathfrak{N}.$$

Notice that any $T \in \mathbf{L}(\mathfrak{H})$ is $\overline{\text{ran}} T_I$ -quasi-selfadjoint and an arbitrary selfadjoint operator is quasi-selfadjoint and $\mathfrak{N}$ -quasi-selfadjoint for each subspace $\mathfrak{N}$ in $\mathfrak{H}$. Moreover, from Definition 4.4 it follows that T is $\mathfrak{N}$ -quasi-selfadjoint if and only if

$$(4.2) \quad T_I = P_{\mathfrak{N}} T_I P_{\mathfrak{N}}.$$

Hence, for every $f, g \in \mathfrak{N}$,

$$\begin{aligned} ((M_{T,\mathfrak{N}}(z) - M_{T,\mathfrak{N}}^*(\xi))f, g) &= (P_{\mathfrak{N}}(T - zI_{\mathfrak{N}})^{-1}f - P_{\mathfrak{N}}(T^* - \bar{\xi}I_{\mathfrak{N}})^{-1}f, g) \\ &= (P_{\mathfrak{N}}(T - zI_{\mathfrak{N}})^{-1}(T^* - T)(T^* - \bar{\xi}I_{\mathfrak{N}})^{-1}f, g) + \\ &\quad + (z - \bar{\xi})(P_{\mathfrak{N}}(T - zI_{\mathfrak{N}})^{-1}(T^* - \bar{\xi}I_{\mathfrak{N}})^{-1}f, g) \\ &= -2i(M_{T,\mathfrak{N}}(z)T_I M_{T,\mathfrak{N}}^*(\xi)f, g) + \\ &\quad + (z - \bar{\xi})(P_{\mathfrak{N}}(T - zI_{\mathfrak{N}})^{-1}(T^* - \bar{\xi}I_{\mathfrak{N}})^{-1}f, g). \end{aligned}$$

Therefore, we obtain the useful formula

$$(4.3) \quad M_{T,\mathfrak{N}}(z) - M_{T,\mathfrak{N}}^*(\xi) + 2iM_{T,\mathfrak{N}}(z)T_I M_{T,\mathfrak{N}}^*(\xi) = (z - \bar{\xi})P_{\mathfrak{N}}(T - zI_{\mathfrak{H}})^{-1}(T^* - \bar{\xi}I_{\mathfrak{H}})^{-1}|_{\mathfrak{N}},$$

where $z, \xi \in \rho(T)$.

Remark 4.5. Let T be an $\mathfrak{N}$ -quasi-selfadjoint operator. Then the operators $\mathcal{A}$ and $\mathcal{A}'$ defined by

$$\mathcal{A} := T|_{\ker T_I}, \quad \mathcal{A}' := T|_{\mathcal{H}},$$

are Hermitian, the operator A in the right lower corner of the matrix representation (4.4) is equal to $P_{\mathcal{H}}\mathcal{A}'$, and

$$\mathcal{A}' \subseteq \mathcal{A} \subset T, \quad \mathcal{A}' \subseteq \mathcal{A} \subset T^*.$$

The latter fact motivates the notion of a quasi-selfadjoint operator.

The block matrix representation (3.3) of an $\mathfrak{N}$ -quasi-selfadjoint operator T can be specified to

$$(4.4) \quad T = \begin{pmatrix} B & C^* \\ C & A \end{pmatrix} : \begin{array}{c} \oplus \\ \mathcal{H} \end{array} \rightarrow \begin{array}{c} \oplus \\ \mathcal{H} \end{array},$$

where A is a bounded selfadjoint operator in $\mathcal{H}$.

For the $\mathfrak{N}$ -Weyl function $W_{T,\mathfrak{N}}$ of an $\mathfrak{N}$ -quasi-selfadjoint operator T , from (3.4) and (3.6) we get

$$(4.5) \quad M_{T,\mathfrak{N}}(z) = -I_{\mathfrak{N}}z^{-1} - Bz^{-2} - (B^2 + C^*C)z^{-3} + o(z^{-3}), \quad z \rightarrow \infty,$$

and

$$(4.6) \quad C^*C = - \lim_{z \rightarrow \infty} z^3(M_{T,\mathfrak{N}}(z) + I_{\mathfrak{N}}z^{-1} + Bz^{-2}) - B^2.$$

Proposition 4.6. *Any $\mathfrak{N}$ -quasi-selfadjoint operator in $\mathfrak{H}$ is $\mathfrak{N}$ -simple if and only if its real part is $\mathfrak{N}$ -simple.*

Proof. If $\mathfrak{N} = \mathfrak{H}$ there is nothing to prove. If $\mathfrak{N} \neq \mathfrak{H}$, then the assertion follows from Proposition 4.2. ■

Corollary 4.7. *If T is an $\mathfrak{N}$ -quasi-selfadjoint operator in $\mathfrak{H}$, then the identity*

$$(4.7) \quad R_T(z) = R_{T_R}(z) - iR_{T_R}(z)T_I(I_{\mathfrak{N}} + iP_{\mathfrak{N}}R_{T_R}(z)T_I|_{\mathfrak{N}})^{-1}P_{\mathfrak{N}}R_{T_R}(z)$$

holds for $z \in \rho(T) \cap \rho(T_R)$.

Proof. The result follows from (4.2) and Corollary 3.6. ■

Recall [20] that a bounded operator T in $\mathfrak{H}$ is called *completely non-selfadjoint* if there is no non-zero reducing subspace of T in which T is a selfadjoint operator. The following results are well-known [20, Theorem 1.2]:

- 1) *the subspaces $\mathfrak{H}_0 = \overline{\text{span}} \{T^n T_I \mathfrak{H} : n \in \mathbb{N}_0\}$ and $\mathfrak{H}' = \mathfrak{H} \ominus \mathfrak{H}_0$ reduce T , $T|_{\mathfrak{H}'}$ is selfadjoint, and $T|_{\mathfrak{H}_0}$ is completely non-selfadjoint,*
- 2) *the operator T is completely non-selfadjoint if and only if*

$$(4.8) \quad \overline{\text{span}} \{T^n T_I \mathfrak{H}, n \in \mathbb{N}_0\} = \mathfrak{H}.$$

Therefore, Proposition 4.6 yields the equivalence of the following statements:

- (1) T is completely non-selfadjoint;
- (2) T is $\overline{\text{ran}} T_I$ -simple;
- (3) T_R is $\overline{\text{ran}} T_I$ -simple.

Now we state a certain converse of Proposition 3.4 (cf. [11, Theorem 6.2]).

Theorem 4.8. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces and $\mathfrak{N} \subseteq \mathfrak{H}_1 \cap \mathfrak{H}_2$ be a subspace. Let T_k , $k = 1, 2$, be $\mathfrak{N}$ -simple and $\mathfrak{N}$ -quasi-selfadjoint operators. If $M_{T_1, \mathfrak{N}} = M_{T_2, \mathfrak{N}}$ in some neighborhood of infinity, then T_1 and T_2 are $\mathfrak{N}$ -unitarily equivalent.*

Proof. Assume

$$M_{T_1, \mathfrak{N}}(z) = M_{T_2, \mathfrak{N}}(z), \quad |z| > r,$$

where r is some positive number. Let $P_{\mathfrak{N}}^{(k)}$ be the orthogonal projections in $\mathfrak{H}_k$ onto $\mathfrak{N}$, $k = 1, 2$. Then

$$P_{\mathfrak{N}}^{(1)} R_{T_1}(z) \upharpoonright \mathfrak{N} = P_{\mathfrak{N}}^{(2)} R_{T_2}(z) \upharpoonright \mathfrak{N}, \quad |z| > r.$$

Relation (4.5) yields $P_{\mathfrak{N}}^{(1)} T_1 \upharpoonright \mathfrak{N} = P_{\mathfrak{N}}^{(2)} T_1 \upharpoonright \mathfrak{N}$. Hence, from (4.2) and (4.3) we get

$$(4.9) \quad (R_{T_1}(z)f, R_{T_1}(\xi)g)_{\mathfrak{H}_1} = (R_{T_2}(z)f, R_{T_2}(\xi)g)_{\mathfrak{H}_2}, \quad |z|, |\xi| > r, \quad f, g \in \mathfrak{N}.$$

Let

$$\mathcal{L}_k = \text{span} \{R_{T_k}(z)\mathfrak{N}, \quad |z| > r\}, \quad k = 1, 2.$$

Let U be a linear operator on $\mathcal{L}_1$ defined as follows:

$$(4.10) \quad U\left(\sum_j R_{T_1}(z_j)f_j\right) = \sum_j R_{T_2}(z_j)f_j, \quad \{f_j\} \subset \mathfrak{N}.$$

Then $\text{ran } U = \mathcal{L}_2$ and (4.9) shows that U is isometric. Since T_1 and T_2 are $\mathfrak{N}$ -simple, the linear manifolds $\mathcal{L}_1$ and $\mathcal{L}_2$ are dense in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Thus, U admits a unitary continuation on $\mathfrak{H}_1$, which is also denoted by U .

Because $UR_{T_1}(z)f = R_{T_2}(z)f$, one has

$$Uf = - \lim_{z \rightarrow \infty} U(zR_{T_1}(z)f) = - \lim_{z \rightarrow \infty} zR_{T_2}(z)f = f, \quad f \in \mathfrak{N}.$$

Finally, if $\{f_j\} \subset \mathfrak{N}$ and $|z_j| > r$, we get

$$\begin{aligned} UT_1\left(\sum_j R_{T_1}(z_j)f_j\right) &= U\left(\sum_j (f_j + z_j R_{T_1}(z_j)f_j)\right) = \sum_j (f_j + z_j R_{T_2}(z_j)f_j) \\ &= T_2\left(\sum_j R_{T_2}(z_j)f_j\right) = T_2U\left(\sum_j R_{T_1}(z_j)f_j\right), \end{aligned}$$

which yields $UT_1 = T_2U$. ■

5. Classes of holomorphic functions

Let $\mathfrak{H}$ be a Hilbert space. We shall be concerned with $\mathbf{L}(\mathfrak{H})$ -valued holomorphic functions with domains containing a neighborhood of infinity, which may depend on the function. We shall be only interested in the behavior of such functions for sufficiently large $|z|$. Therefore, if it is not indicated otherwise, *the equality $F = G$ of two such functions means that there exists a neighborhood of infinity, where F and G coincide.*

5.1. Herglotz–Nevanlinna functions. An $\mathbf{L}(\mathfrak{H})$ -valued function N is called a Herglotz–Nevanlinna function or an R -function if it is holomorphic in the upper and the lower half-planes, $N^*(z) = N(\bar{z})$, and

$$\frac{N(z) - N^*(z)}{z - \bar{z}} \geq 0$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. An account on scalar Herglotz–Nevanlinna functions can be found in [30]. Many properties of scalar functions can be easily generalized to operator-valued Herglotz–Nevanlinna functions, cf. [20, §4].

Definition 5.1. An $\mathbf{L}(\mathfrak{H})$ -valued Herglotz–Nevanlinna function N is said to belong to the class $\mathbf{N}_{\mathfrak{H}}$ if it is holomorphic at infinity.

If, in addition,

$$\lim_{z \rightarrow \infty} zN(z) = -I_{\mathfrak{H}},$$

then we say that N belongs to the class $\mathbf{N}_{\mathfrak{H}}^0$.

Clearly, for $N \in \mathbf{N}_{\mathfrak{H}}$ the value $N(\infty) := \lim_{z \rightarrow \infty} N(z)$ belongs to $\mathbf{L}(\mathfrak{H})$.

It is well known that every function N of the class $\mathbf{N}_{\mathfrak{H}}$ admits an integral representation of the form

$$(5.1) \quad N(z) = \Gamma + \int_a^b \frac{dG(t)}{t - z}$$

for z in the domain of N . Here Γ is a selfadjoint operator in $\mathfrak{H}$ and G is a bounded nondecreasing and nonnegative $\mathbf{L}(\mathfrak{H})$ -valued function on some bounded interval $[a, b]$ normalized by $G(a) = 0$, cf. [20, §4], [30]. Clearly, $N(\infty) = \Gamma$. Moreover, $N \in \mathbf{N}_{\mathfrak{H}}^0$ if and only if $\Gamma = 0$ and $\int_a^b dG(t) = I_{\mathfrak{H}}$.

The following operator representations are well known.

Theorem 5.2. ([20]) *Let $\mathfrak{N}$ be a Hilbert space and let $N \in \mathbf{N}_{\mathfrak{N}}$. Then there exist a Hilbert space $\mathcal{H}$, a bounded selfadjoint operator A in $\mathcal{H}$ and an operator $C \in \mathbf{L}(\mathfrak{N}, \mathcal{H})$, such that*

$$(5.2) \quad N(z) = N(\infty) + C^* R_A(z) C$$

for z in the domain of N . Moreover, $\mathcal{H}$ and the operators A and C can be chosen in such a way that A is $\overline{\text{ran}} C$ -simple.

It should be mentioned that the right-hand side of (5.2) is defined and holomorphic on $\rho(A)$. From Definition 3.1 and expansion (4.5) it follows that the $\mathfrak{N}$ -Weyl function $M_{T, \mathfrak{N}}$ of a bounded selfadjoint operator T belongs to the class $\mathbf{N}_{\mathfrak{N}}^0$. Due to Naïmark's dilation theorem [20, Theorem 2 of Appendix 1] the converse statement is also true.

Theorem 5.3. ([20]) *If $M \in \mathbf{N}_{\mathfrak{N}}^0$, then there exist a Hilbert space $\mathfrak{H}$ containing $\mathfrak{N}$ as a subspace and a selfadjoint operator T in $\mathfrak{H}$ such that T is $\mathfrak{N}$ -simple and*

$$(5.3) \quad M(z) = P_{\mathfrak{N}} R_T(z) \upharpoonright \mathfrak{N}$$

for z in the domain of M .

Notice that according to Theorem 4.8 the pair $\langle \mathfrak{H}, T \rangle$ is defined up to $\mathfrak{N}$ -unitary equivalence. Thus, we arrive at the following statement.

Theorem 5.4. *An $\mathbf{L}(\mathfrak{N})$ -valued function belongs to the class $\mathbf{N}_{\mathfrak{N}}^0$ if and only if it is the $\mathfrak{N}$ -Weyl function of an $\mathfrak{N}$ -simple selfadjoint operator.*

The next two theorems will be used in Section 6.

Theorem 5.5. *Let $M \in \mathbf{N}_{\mathfrak{N}}^0$. Then the following statements hold true:*

- (1) $M^{-1}(z) \in \mathbf{L}(\mathfrak{N})$ for all z in some neighborhood of infinity;
- (2) if

$$B_0 = - \lim_{z \rightarrow \infty} z^2 (M(z) + I_{\mathfrak{N}} z^{-1}),$$

then the function

$$N(z) := B_0 - z I_{\mathfrak{N}} - M^{-1}(z)$$

belongs to the class $\mathbf{N}_{\mathfrak{N}}$ and $N(\infty) = 0$.

Proof. The statements follow from Theorem 5.4, relation (3.9), and Theorem 3.2. ■

Theorem 5.6. *Let $N \in \mathbf{N}_{\mathfrak{N}}$ and $N(\infty) = 0$. Let*

$$D_0 = - \lim_{z \rightarrow \infty} zN(z), \quad \mathfrak{N}_1 := \overline{\text{ran}} D_0.$$

Then there exists a function $N_1 \in \mathbf{N}_{\mathfrak{N}_1}^0$ such that

$$N = D_0^{1/2} N_1 D_0^{1/2}.$$

Proof. For z in the domain of N the function N admits an integral representation

$$N(z) = \int_a^b \frac{dG(t)}{t - z}.$$

Notice that

$$D_0 = \int_a^b dG(t).$$

Since $0 \leq G(t) \leq D_0$, we obtain

$$G(t) = D_0^{1/2} G_1(t) D_0^{1/2}, \quad t \in [a, b],$$

for some nonnegative and nondecreasing $\mathbf{L}(\mathfrak{N}_1)$ -valued function G_1 such that $G_1(a) = 0$, $\int_a^b dG_1(t) = I_{\mathfrak{N}_1}$. Clearly, the $\mathbf{L}(\mathfrak{N}_1)$ -valued function

$$N_1(z) := \int_a^b \frac{dG_1(t)}{t - z}, \quad z \in \mathbb{C} \setminus [a, b],$$

belongs to the class $\mathbf{N}_{\mathfrak{N}_1}^0$ and $N = D_0^{1/2} N_1 D_0^{1/2}$. ■

Remark 5.7. It is obvious that the function N_1 is unique in accordance with the convention mentioned at the beginning of this section.

5.2. The class $\mathbf{M}_{\mathfrak{N}}^0$. Before we define the class $\mathbf{M}_{\mathfrak{N}}^0$, we establish the main properties of the $\mathfrak{N}$ -Weyl function of an $\mathfrak{N}$ -quasi-selfadjoint operator.

Theorem 5.8. *Let $M_{T,\mathfrak{N}}$ be the $\mathfrak{N}$ -Weyl function of an $\mathfrak{N}$ -quasi-selfadjoint operator $T \in \mathbf{L}(\mathfrak{H})$. Put $B = P_{\mathfrak{N}}T|_{\mathfrak{N}}$. Then*

(1) *$M_{T,\mathfrak{N}}$ has the asymptotic expansion:*

$$(5.4) \quad M_{T,\mathfrak{N}}(z) = -I_{\mathfrak{N}}z^{-1} - Bz^{-2} + o(z^{-2}), \quad z \rightarrow \infty;$$

(2) *$M_{T,\mathfrak{N}}^{-1}(z) \in \mathbf{L}(\mathfrak{N})$ for all $|z| > \|T\|$;*

(3) *the function*

$$N(z) := -M_{T,\mathfrak{N}}^{-1}(z) + B - zI_{\mathfrak{N}}, \quad |z| > \|T\|,$$

belongs to $\mathbf{N}_{\mathfrak{N}}$ and admits a holomorphic continuation on the domain $\mathbb{C} \setminus [\alpha, \beta]$, where

$$(5.5) \quad \begin{aligned} \alpha &:= \inf\{(Th, h), h \in \mathcal{H}, \|h\| = 1\}, \\ \beta &:= \sup\{(Th, h), h \in \mathcal{H}, \|h\| = 1\}, \end{aligned} \quad \mathcal{H} = \mathfrak{H} \ominus \mathfrak{N};$$

(4) *the function*

$$(5.6) \quad N_0(z) := (I_{\mathfrak{N}} - iM_{T,\mathfrak{N}}(z)B_I)^{-1}M_{T,\mathfrak{N}}(z), \quad |z| > \|T\|,$$

belongs to $\mathbf{N}_{\mathfrak{N}}^0$ and admits a holomorphic continuation on the domain $\mathbb{C} \setminus [a, b]$, where

$$(5.7) \quad a := \inf\{(T_R h, h), h \in \mathfrak{H}, \|h\| = 1\}, \quad b := \sup\{(T_R h, h), h \in \mathfrak{H}, \|h\| = 1\};$$

moreover, $N_0 = M_{T_R, \mathfrak{N}}$, i.e., N_0 is the $\mathfrak{N}$ -Weyl function of the real part of T ;

(5) *the function of two variables*

$$(5.8) \quad G(z, \xi) = \begin{cases} \frac{M_{T,\mathfrak{N}}(z) - M_{T,\mathfrak{N}}(\bar{\xi})}{z - \bar{\xi}} M_{T,\mathfrak{N}}^{-1}(\bar{\xi}) M_{T,\mathfrak{N}}^*(\xi), & \xi \neq \bar{z}, \\ \frac{d(M_{T,\mathfrak{N}}(z))}{dz} M_{T,\mathfrak{N}}^{-1}(z) M_{T,\mathfrak{N}}^*(\bar{z}), & \xi = \bar{z} \end{cases}$$

is a nonnegative kernel (see [15], [34]) for $|z|, |\xi| > \|T\|$, i.e.,

$$\sum_{i,j=1}^n (G(\lambda_j, \lambda_i) f_i, f_j)_{\mathfrak{N}} \geq 0$$

for every natural number n and any choice of points $\{\lambda_i\}_{i=1}^n$ and vectors $\{f_i\}_{i=1}^n \subset \mathfrak{N}$.

Proof. (1): The asymptotic expansion (5.4) follows from (4.5).

(2) and (3): The operator T has the block operator matrix form (4.4). From the Schur–Frobenius formula (3.8) we get that $M_{T,\mathfrak{N}}^{-1}(z) \in \mathbf{L}(\mathfrak{N})$ and

$$M_{T,\mathfrak{N}}^{-1}(z) = -zI_{\mathfrak{N}} - V(z), \quad |z| > \|T\|,$$

where $V(z) = -B + C^*R_A(z)C$. It follows that

$$N(z) = C^*R_A(z)C, \quad |z| > \|T\|.$$

Because $A = P_{\mathcal{H}}T|_{\mathcal{H}}$ is a selfadjoint operator, the function N belongs to the class $\mathbf{N}_{\mathfrak{N}}$ and admits a holomorphic continuation on the domain $\mathbb{C} \setminus [\alpha, \beta]$, where the numbers α and β are defined by (5.5).

(4): Let $M_{T_R,\mathfrak{N}}$ be the $\mathfrak{N}$ -Weyl function of the operator T_R . Since T_R is selfadjoint, by Definition 3.1 and Definition 5.1 it belongs to the class $\mathbf{N}_{\mathfrak{N}}^0$ and has a holomorphic continuation on the domain $\mathbb{C} \setminus [a, b]$, where the numbers a and b are given by (5.7). From the resolvent formula (4.7) we obtain

$$P_{\mathfrak{N}}R_T(z)|_{\mathfrak{N}} = (P_{\mathfrak{N}}R_{T_R}(z) - iP_{\mathfrak{N}}R_{T_R}(z)B_I(I_{\mathfrak{N}} + iM_{T_R,\mathfrak{N}}(z)B_I)^{-1}P_{\mathfrak{N}}R_{T_R}(z))|_{\mathfrak{N}},$$

where $|z| > \|T\|$. It follows

$$\begin{aligned} M_{T,\mathfrak{N}}(z) &= M_{T_R,\mathfrak{N}}(z) - iM_{T_R,\mathfrak{N}}(z)B_I(I_{\mathfrak{N}} + iM_{T_R,\mathfrak{N}}(z)B_I)^{-1}M_{T_R,\mathfrak{N}}(z) \\ &= (I_{\mathfrak{N}} + iM_{T_R,\mathfrak{N}}(z)B_I)^{-1}M_{T_R,\mathfrak{N}}(z). \end{aligned}$$

Hence, $(I_{\mathfrak{N}} - iM_{T,\mathfrak{N}}(z)B_I)^{-1}M_{T,\mathfrak{N}}(z) = M_{T_R,\mathfrak{N}}(z)$, $|z| > \|T\|$.

(5): Notice that from (4) it follows that

$$M_{T,\mathfrak{N}}^{-1}(z) = M_{T_R,\mathfrak{N}}^{-1}(z) + iB_I, \quad |z| > \|T\|.$$

Furthermore, for $\xi \neq \bar{z}$ we have

$$\begin{aligned} & \frac{M_{T,\mathfrak{N}}(z) - M_{T,\mathfrak{N}}^*(\xi) + 2iM_{T,\mathfrak{N}}(z)B_IM_{T,\mathfrak{N}}^*(\xi)}{z - \bar{\xi}} \\ &= M_{T,\mathfrak{N}}(z) \frac{M_{T,\mathfrak{N}}^{*-1}(\xi) - M_{T,\mathfrak{N}}^{-1}(z) + 2iB_IM_{T,\mathfrak{N}}^*(\xi)}{z - \bar{\xi}} \\ &= -M_{T,\mathfrak{N}}(z) \frac{M_{T_R,\mathfrak{N}}^{-1}(z) - M_{T_R,\mathfrak{N}}^{-1}(\bar{\xi})}{z - \bar{\xi}} M_{T,\mathfrak{N}}^*(\xi) \\ &= -M_{T,\mathfrak{N}}(z) \frac{M_{T,\mathfrak{N}}^{-1}(z) - M_{T,\mathfrak{N}}^{-1}(\bar{\xi})}{z - \bar{\xi}} M_{T,\mathfrak{N}}^*(\xi) \\ &= \frac{M_{T,\mathfrak{N}}(z) - M_{T,\mathfrak{N}}(\bar{\xi})}{z - \bar{\xi}} M_{T,\mathfrak{N}}^{-1}(\bar{\xi}) M_{T,\mathfrak{N}}^*(\xi). \end{aligned}$$

Hence, (4.3) yields

$$\sum_{i,j=1}^n (G(\lambda_j, \lambda_i) f_i, f_j)_{\mathfrak{H}} = \left\| \sum_{j=1}^n (T^* - \bar{\lambda}_j I_{\mathfrak{H}})^{-1} f_j \right\|^2 \geq 0$$

for every choice of points $\{\lambda_i\}_{i=1}^n$, $|\lambda_i| > \|T\|$, $i = 1, \dots, n$, and vectors $\{f_i\}_{i=1}^n \subset \mathfrak{H}$, where $G(z, \xi)$ is defined by (5.8). ■

Remark 5.9. If the operator T is selfadjoint, then $M_{T, \mathfrak{H}}$ belongs to the class $\mathbf{N}_{\mathfrak{H}}^0$ and the kernel $G(z, \xi)$ takes the well known form

$$G(z, \xi) = \begin{cases} \frac{M_{T, \mathfrak{H}}(z) - M_{T, \mathfrak{H}}^*(\xi)}{z - \bar{\xi}}, & \xi \neq \bar{z}, \\ \frac{d(M_{T, \mathfrak{H}}(z))}{dz}, & \xi = \bar{z}. \end{cases}$$

If $\dim \mathfrak{H} < \infty$, then Theorem 3.2 implies that the $\mathfrak{H}$ -Weyl function of an $\mathfrak{H}$ -quasi-selfadjoint operator T is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ except, perhaps, a set of poles in the closed disc $\{z \in \mathbb{C} : |z| \leq \|T\|\}$ whose accumulation points are located on the real axis (see [20, Theorem 9.1]).

The preceding Theorem 5.8 suggests the introduction of the following class of operator-valued holomorphic functions.

Definition 5.10. Let $\mathfrak{H}$ be a Hilbert space. We will say that an $\mathbf{L}(\mathfrak{H})$ -valued function M belongs to the class $\mathbf{M}_{\mathfrak{H}}^0$ if the following conditions are satisfied:

- (1) M is holomorphic in some neighborhood of infinity,
- (2) M has the asymptotic expansion

$$M(z) = -I_{\mathfrak{H}} z^{-1} - B z^{-2} + o(z^{-2}), \quad z \rightarrow \infty,$$

where $B \in \mathbf{L}(\mathfrak{H})$,

- (3) the function

$$M_0(z) := (I_{\mathfrak{H}} - iM(z)B_I)^{-1}M(z),$$

which is defined for all z in some neighborhood of infinity, has a holomorphic continuation to a function of the class $\mathbf{N}_{\mathfrak{H}}^0$.

Notice that if the operator B of condition (2) is selfadjoint, then $M \in \mathbf{N}_{\mathfrak{H}}^0$. Therefore, the class $\mathbf{N}_{\mathfrak{H}}^0$ is contained in the class $\mathbf{M}_{\mathfrak{H}}^0$. By Theorem 5.8, the $\mathfrak{H}$ -Weyl function of an $\mathfrak{H}$ -quasi-selfadjoint operator belongs to $\mathbf{M}_{\mathfrak{H}}^0$. The next theorem deals with a converse statement.

Theorem 5.11. *If M belongs to the class $\mathbf{M}_{\mathfrak{N}}^0$, then there exist a Hilbert space $\mathfrak{H}$ containing $\mathfrak{N}$ as a subspace and an $\mathfrak{N}$ -simple and $\mathfrak{N}$ -quasi-selfadjoint operator T in $\mathfrak{H}$ such that*

$$M_{T,\mathfrak{N}} = M.$$

Proof. Define the function M_0 according to (3) of Definition 5.10. Since M_0 has a holomorphic continuation to a function of the class $\mathbf{N}_{\mathfrak{N}}^0$, by Theorem 5.3 there exist a Hilbert space $\mathfrak{H}$ containing $\mathfrak{N}$ as a subspace and an $\mathfrak{N}$ -simple selfadjoint operator T_0 in $\mathfrak{H}$ such that M_0 is its $\mathfrak{N}$ -Weyl function. It is easy to see that M_0 has an asymptotic expansion

$$M_0(z) = -I_{\mathfrak{N}}z^{-1} - B_Rz^{-2} + o(z^{-2}), \quad z \rightarrow \infty,$$

which yields $B_R = P_{\mathfrak{N}}T_0|_{\mathfrak{N}}$. Set $T = T_0 + iB_IP_{\mathfrak{N}}$. Then T is quasi-selfadjoint, $T_R = T_0$, and $\mathfrak{N} \supseteq \overline{\text{ran}}T_I$. By Proposition 4.6, T is also $\mathfrak{N}$ -simple. The resolvent formula (4.7) gives

$$P_{\mathfrak{N}}R_T(z)|_{\mathfrak{N}} = P_{\mathfrak{N}}(R_{T_0}(z) - iR_{T_0}(z)T_I(I_{\mathfrak{N}} + iP_{\mathfrak{N}}R_{T_0}(z)T_I|_{\mathfrak{N}})^{-1}P_{\mathfrak{N}}R_{T_0}(z))|_{\mathfrak{N}},$$

hence, for z in a neighborhood of infinity,

$$\begin{aligned} M_{T,\mathfrak{N}}(z) &= M_{T_0,\mathfrak{N}}(z) - iM_{T_0,\mathfrak{N}}(z)B_I(I_{\mathfrak{N}} + iM_{T_0,\mathfrak{N}}(z)B_I)^{-1}M_{T_0,\mathfrak{N}}(z) \\ &= (I_{\mathfrak{N}} + iM_0(z)B_I)^{-1}M_0(z) \end{aligned}$$

since $T_I = B_IP_{\mathfrak{N}}$. Taking into account condition (3) of Definition 5.10, we obtain the equality $M_{T,\mathfrak{N}} = M$. ■

Notice that by Theorem 4.8 the $\mathfrak{N}$ -simple and $\mathfrak{N}$ -quasi-selfadjoint operator T of Theorem 5.11 is unique up to $\mathfrak{N}$ -unitary equivalence.

Proposition 5.12. ([11]) *Suppose $M \in \mathbf{M}_{\mathfrak{N}}^0$ and let $B = -\lim_{z \rightarrow \infty} z^2(M(z) + I_{\mathfrak{N}}z^{-1})$. Then the following conditions are equivalent:*

- (i) *the Hilbert space $\mathfrak{H}$ of Theorem 5.11 coincides with $\mathfrak{N}$;*
- (ii) *the function $z \rightarrow M^{-1}(z) + zI - B$ is identically zero;*
- (iii)

$$-\lim_{z \rightarrow \infty} z^3(M(z) + I_{\mathfrak{N}}z^{-1} + Bz^{-2}) = B^2.$$

Proof. By Theorem 5.11 the function M is the $\mathfrak{N}$ -Weyl function of some $\mathfrak{N}$ -simple and $\mathfrak{N}$ -quasi-selfadjoint operator T in a Hilbert space $\mathfrak{H}$.

$$(i) \iff (ii): \mathfrak{H} = \mathfrak{N} \iff M(z) = R_T(z) \iff M^{-1}(z) + zI_{\mathfrak{N}} = T = B.$$

(i) $\Rightarrow$ (iii): If $\mathfrak{H} = \mathfrak{N}$, then

$$M(z) = -I_{\mathfrak{N}}z^{-1} - Tz^{-2} - T^2z^{-3} + o(z^{-3}), \quad z \rightarrow \infty.$$

Hence,

$$\begin{aligned} B = T &= -\lim_{z \rightarrow \infty} z^2 (M(z) + I_{\mathfrak{N}}z^{-1}), \\ T^2 &= -\lim_{z \rightarrow \infty} z^3 (M(z) + I_{\mathfrak{N}}z^{-1} + Tz^{-2}) = B^2. \end{aligned}$$

(iii) $\Rightarrow$ (i): If $B^2 = -\lim_{z \rightarrow \infty} z^3 (M(z) + I_{\mathfrak{N}}z^{-1} + Bz^{-2})$, then

$$B = P_{\mathfrak{N}}T|_{\mathfrak{N}} \quad \text{and} \quad B^2 = P_{\mathfrak{N}}T^2|_{\mathfrak{N}}.$$

Because T is of the matrix form (4.4), we get $C = P_{\mathfrak{N}}T|_{\mathcal{H}} = 0$. Hence, $\mathfrak{N}$ reduces T . Since T is $\mathfrak{N}$ -simple, we get $\mathfrak{H} = \mathfrak{N}$. ■

Let $M \in \mathbf{M}_{\mathfrak{N}}^0$ and let $X \in \mathbf{L}(\mathfrak{N})$. Since $\lim_{z \rightarrow \infty} M(z) = 0$ with respect to the operator norm topology, there exists a neighborhood of infinity, where the function $\Lambda = \Phi_X(M)$ given by the fractional linear transformation

$$(5.9) \quad \Lambda(z) = M(z)(I_{\mathfrak{N}} + XM(z))^{-1} = (I_{\mathfrak{N}} + M(z)X)^{-1}M(z)$$

is defined and holomorphic. Moreover, $\lim_{z \rightarrow \infty} z\Lambda(z) = -I_{\mathfrak{N}}$ and

$$(5.10) \quad \Lambda^{-1}(z) = M^{-1}(z) + X$$

in a neighborhood of infinity. The inverse transform is equal to

$$M = \Phi_{-X}(\Lambda).$$

Notice that from Definition 5.10 it follows that for any $M \in \mathbf{M}_{\mathfrak{N}}^0$ there exist a function $N \in \mathbf{N}_{\mathfrak{N}}^0$ and an operator $X \in \mathbf{L}(\mathfrak{N})$ such that $M = \Phi_X(N)$.

Proposition 5.13. (1) *For any $X \in \mathbf{L}(\mathfrak{N})$ the transformation Φ_X maps $\mathbf{M}_{\mathfrak{N}}^0$ onto $\mathbf{M}_{\mathfrak{N}}^0$. In addition, the set*

$$\{\Phi_X, X \in \mathbf{L}(\mathfrak{N})\}$$

forms a group of fractional linear transformations of $\mathbf{M}_{\mathfrak{N}}^0$ isomorphic to the additive group of $\mathbf{L}(\mathfrak{N})$.

(2) *The transformation Φ_X maps $\mathbf{N}_{\mathfrak{N}}^0$ into $\mathbf{N}_{\mathfrak{N}}^0$ if and only if the operator X is selfadjoint.*

Proof. From (5.9) we have the identity

$$\begin{aligned} \Lambda(z) - \Lambda^*(z) &= (I_{\mathfrak{H}} + N(z)X)^{-1}(N(z) - N^*(z) + N(z)(X - X^*)N^*(z)) \times \\ &\quad \times (I_{\mathfrak{H}} + X^*N^*(z))^{-1}, \end{aligned}$$

which holds for z in some neighborhood of infinity. Thus, if $X = X^*$ and $N \in \mathbf{N}_{\mathfrak{H}}^0$, then, clearly, $\Lambda \in \mathbf{N}_{\mathfrak{H}}^0$. Conversely, if $N, \Lambda \in \mathbf{N}_{\mathfrak{H}}^0$, then the equalities $N^*(z) = N(z)$ and $\Lambda^*(z) = \Lambda(z)$ and the invertibility of $N(z)$ for all real z in a neighborhood of infinity yield $X = X^*$.

In order to prove statement (1) we note that if $X, Y \in \mathbf{L}(\mathfrak{H})$, then

$$\begin{aligned} \Phi_Y(\Phi_X(M)) &= M(I_{\mathfrak{H}} + XM)^{-1}(I_{\mathfrak{H}} + YM(I_{\mathfrak{H}} + XM)^{-1})^{-1} \\ &= M(I_{\mathfrak{H}} + (X + Y)M)^{-1} = \Phi_{X+Y}(M). \end{aligned}$$

Let $M \in \mathbf{M}_{\mathfrak{H}}^0$, and let $\lim_{z \rightarrow \infty} z^2(M(z) + I_{\mathfrak{H}}z^{-1}) = -B$. Then for $\Lambda = \Phi_X(M)$, we get

$$(5.11) \quad \lim_{z \rightarrow \infty} z^2(\Lambda(z) + I_{\mathfrak{H}}z^{-1}) = -B - X.$$

It follows that

$$\Phi_{-iB_I - iX_I}(\Lambda) = \Phi_{-iX_I + X - iB_I}(M) = \Phi_{X_R}(\Phi_{-iB_I}(M)).$$

By Definition 5.10, the function $\Phi_{-iB_I}(M)$ belongs to the class $\mathbf{N}_{\mathfrak{H}}^0$. From statement (2) we obtain that $\Phi_{-iB_I - iX_I}(\Lambda)$ also belongs to the class $\mathbf{N}_{\mathfrak{H}}^0$. Therefore, $\Lambda = \Phi_X(M)$ belongs to the class $\mathbf{M}_{\mathfrak{H}}^0$. ■

Remark 5.14. The Weyl functions of quasi-selfadjoint contractions and of quasi-selfadjoint operators with rank one imaginary part have been investigated in [11] and [13], respectively.

6. The Schur transformation of the class $\mathbf{M}_{\mathfrak{H}}^0$

Theorem 6.1. *Let $\mathfrak{H}$ be a Hilbert space and $M \in \mathbf{M}_{\mathfrak{H}}^0$. Define the operators*

$$(6.1) \quad B = - \lim_{z \rightarrow \infty} z^2(M(z) + I_{\mathfrak{H}}z^{-1})$$

and

$$(6.2) \quad D_0 = - \lim_{z \rightarrow \infty} z^3(M(z) + Iz^{-1} + Bz^{-2}) - B^2.$$

Then $D_0 \geq 0$ and the following statements hold true:

(1) the function

$$(6.3) \quad z \mapsto N(z) = B - zI_{\mathfrak{N}} - M^{-1}(z),$$

initially defined in a neighborhood of infinity, has a continuation to a function from the class $\mathbf{N}_{\mathfrak{N}}$ and satisfies

$$(6.4) \quad -\lim_{z \rightarrow \infty} zN(z) = D_0;$$

(2) for $\mathfrak{N}_1 = \overline{\text{ran}} D_0$, there exists a unique function M_1 of the class $\mathbf{N}_{\mathfrak{N}_1}^0$ such that

$$(6.5) \quad D_0^{1/2} M_1 D_0^{1/2} = N;$$

(3) the following conditions are equivalent:

- (a) the function N is identically zero,
- (b) $\mathfrak{N}_1 = \{0\}$,
- (c) $D_0 = 0$,
- (d) $-\lim_{z \rightarrow \infty} z^3 (M(z) + I_{\mathfrak{N}} z^{-1} + B z^{-2}) = B^2$.

Proof. By Theorem 5.11 the function M can be realized as the $\mathfrak{N}$ -Weyl function

$$M(z) = P_{\mathfrak{N}} R_T(z) \upharpoonright \mathfrak{N}, \quad |z| > \|T\|,$$

of an $\mathfrak{N}$ -quasi-selfadjoint and $\mathfrak{N}$ -simple operator T in some Hilbert space $\mathfrak{H}$ containing $\mathfrak{N}$ as a subspace. According to Proposition 5.12, the case $\mathfrak{H} = \mathfrak{N}$ occurs if and only if one of the equivalent conditions (a), (b), (c), (d) is satisfied

Suppose $\mathfrak{H} \neq \mathfrak{N}$. Represent T in the block-matrix form (4.4) with respect to the decomposition $\mathfrak{H} = \mathfrak{N} \oplus \mathcal{H}$. Since $A = P_{\mathcal{H}} T \upharpoonright \mathcal{H}$, we have $\sigma(A) \subseteq [-\|T\|, \|T\|]$. From Theorem 3.2 and formulas (3.8), (3.7), (4.5), and (4.6) it follows that

$$D_0 = C^* C \geq 0,$$

and

$$M^{-1}(z) = B - zI_{\mathfrak{N}} - C^* R_A(z) C, \quad |z| > \|T\|.$$

Hence, for $z \in \rho(A)$ one has

$$N(z) = C^* R_A(z) C,$$

which implies that N belongs to the class $\mathbf{N}_{\mathfrak{N}}$. To obtain statement (2) we apply Theorems 5.6 and 5.5. ■

Remark 6.2. Since $C^*C = D_0$, there exists a unitary operator V_0 from $\mathfrak{N}_1$ onto $\mathfrak{M} := \overline{\text{ran}} C \subseteq \mathcal{H}$ such that $C = V_0 D_0^{1/2}$. Therefore,

$$C^* R_A(z) C = D_0^{1/2} V_0^* P_{\mathfrak{M}} R_A(z) V_0 D_0^{1/2} = V_0^* M_{A, \mathfrak{M}}(z) V_0, \quad z \in \rho(A).$$

It follows that $M_1 = V_0^* M_{A, \mathfrak{M}} V_0$.

Set

$$\mathcal{H}_1 = \mathcal{H} \ominus \mathfrak{M}, \quad \mathfrak{H}_1 = \mathfrak{N}_1 \oplus \mathcal{H}_1,$$

and consider a unitary operator

$$\mathcal{V}_0 := \begin{pmatrix} V_0 & 0 \\ 0 & I_{\mathcal{H}_1} \end{pmatrix} : \begin{matrix} \mathfrak{N}_1 & \mathfrak{M} \\ \oplus & \rightarrow \oplus \\ \mathcal{H}_1 & \mathcal{H}_1 \end{matrix}.$$

Define a selfadjoint operator $T^{(1)}$ in $\mathfrak{H}_1$ by the equality

$$T^{(1)} = \mathcal{V}_0^* A \mathcal{V}_0.$$

Clearly, $T^{(1)}$ is unitarily equivalent to A and $M_{T^{(1)}, \mathfrak{N}_1} = V_0^* M_{A, \mathfrak{M}} V_0$. Thus,

$$M_1 = M_{T^{(1)}, \mathfrak{N}_1},$$

i.e., the function M_1 is equal to the $\mathfrak{N}_1$ -Weyl function of $T^{(1)}$ and its domain of analyticity contains the neighborhood of infinity $U_{\|T^{(1)}\|} = \{z : |z| > \|T^{(1)}\|\}$. Since $\|T^{(1)}\| = \|A\| \leq \|T\|$, both functions M and M_1 are holomorphic in the neighborhood of infinity

$$U_{\|T\|} = \{z \in \mathbb{C} : |z| > \|T\|\}.$$

In addition

$$\|B\| \leq \|T\|, \quad \|C\| = \|D_0^{1/2}\| \leq \|T\|.$$

Definition 6.3. The function M_1 appearing in equality (6.5) is called *the Schur transformation of the function M* .

If D_0 is positive definite, then $\mathfrak{N}_1 = \mathfrak{N}$ and

$$M_1(z) = D_0^{-1/2} (B - zI_{\mathfrak{N}} - M^{-1}(z)) D_0^{-1/2} \in \mathbf{N}_{\mathfrak{N}}^0.$$

If M is a scalar function (i.e., $\dim \mathfrak{N} = 1$), then the Schur transformation takes on the classical form [1], [29], [5]

$$M_1(z) = \frac{1}{a_0^2} \left(b - z - \frac{1}{M(z)} \right),$$

where

$$a_0^2 = d_0 = - \lim_{z \rightarrow \infty} z \left(b - z - \frac{1}{M(z)} \right) = - \lim_{z \rightarrow \infty} z^3 \left(M(z) + \frac{1}{z} + \frac{b}{z^2} \right) - b^2 \neq 0.$$

Remark 6.4. Let $M \in \mathbf{M}_{\mathfrak{N}}^0$, $X \in \mathbf{L}(\mathfrak{N})$, and let $\Lambda = \Phi_X(M)$ be the fractional linear transformation of M according to (5.9). From (5.10), (5.11), and (6.4) it follows that the Schur transformations of M and Λ coincide in a neighborhood of infinity.

Proposition 6.5. *Let $M \in \mathbf{N}_{\mathfrak{N}}^0$. If M is an odd function in some neighborhood of infinity, then $B = 0$ and the Schur transformation of M is an odd function as well.*

Proof. If M is odd, (6.1) yields $B = 0$. Therefore, N is odd by (6.3) and, hence, M_1 is odd by (6.5). ■

Let $M_0 \in \mathbf{M}_{\mathfrak{N}}^0$. Define the operators B_0 and D_0 by (6.1) and (6.4), respectively. Put $\mathfrak{N}_1 = \overline{\text{ran}} D_0$. Note that

$$M_0(z) = -I_{\mathfrak{N}_0} z^{-1} - B_0 z^{-2} - (B_0^2 + D_0) z^{-3} + o(z^{-3}), \quad z \rightarrow \infty.$$

Define the $\mathbf{L}(\mathfrak{N}_1)$ -valued Schur transformation M_1 of the function M_0 by the relation

$$D_0^{1/2} M_1(z) D_0^{1/2} = B_0 - z I_{\mathfrak{N}_0} - M_0^{-1}(z).$$

According to Theorem 6.1 the function M_1 belongs to the class $\mathbf{N}_{\mathfrak{N}_1}^0$. Now we apply the preceding construction to the function M_1 instead of M_0 and continue this procedure. If $\mathfrak{N}_{N+1} = \{0\}$ ($\iff D_N = 0$), then the process terminates at the N -th step. Otherwise the process is infinite. This way we get

- (1) a finite or infinite chain of nontrivial subspaces

$$\mathfrak{N} = \mathfrak{N}_0 \supseteq \mathfrak{N}_1 \supseteq \mathfrak{N}_2 \supseteq \cdots,$$

- (2) bounded selfadjoint operators $B_k \in \mathbf{L}(\mathfrak{N}_k)$,
 (3) bounded selfadjoint nonnegative operators $D_k \in \mathbf{L}(\mathfrak{N}_k)$, such that $\mathfrak{N}_{k+1} = \overline{\text{ran}} D_k$,
 (4) functions $M_k \in \mathbf{N}_{\mathfrak{N}_k}^0$, which are defined in a common neighborhood of infinity U , have the asymptotic behavior

$$M_k(z) = -I_{\mathfrak{N}_k} z^{-1} - B_k z^{-2} - (B_k^2 + D_k) z^{-3} + o(z^{-3}), \quad z \rightarrow \infty,$$

and M_{k+1} is the Schur transformation of M_k , i.e.,

$$M_k^{-1}(z) = B_k - z I_{\mathfrak{N}_k} - D_k^{1/2} M_{k+1}(z) D_k^{1/2}, \quad z \in U,$$

for all k .

Moreover, from Remark 6.2 it follows

$$\sup_{k \geq 0} \{ \|B_k\| + \|D_k\| \} < \infty.$$

Definition 6.6. Let $M_0 \in \mathbf{M}_{\mathfrak{N}}^0$. The sequences of subspaces

$$\mathfrak{N} = \mathfrak{N}_0 \supseteq \mathfrak{N}_1 \supseteq \mathfrak{N}_2 \supseteq \cdots,$$

and of operators $\{B_k\}_{k \geq 0}$ and $\{D_k\}_{k \geq 0}$, described above, are said to be the sequences of Schur parameters of M_0 .

Notice that the operators B_k are selfadjoint for $k \geq 1$, the operators D_k are nonnegative for all $k \geq 0$. All operators B_k , $k \geq 0$, are selfadjoint if and only if M_0 belongs to the class $\mathbf{N}_{\mathfrak{N}}^0$.

Theorem 6.7. Let M_0 and $\widetilde{M}_0$ be from the class $\mathbf{M}_{\mathfrak{N}}^0$. If the Schur parameters of M_0 and $\widetilde{M}_0$ coincide, then $M_0 = \widetilde{M}_0$.

Proof. Let

$$\mathfrak{N} = \mathfrak{N}_0 \supseteq \mathfrak{N}_1 \supseteq \mathfrak{N}_2 \supseteq \cdots, \quad \{B_k \in \mathbf{L}(\mathfrak{N}_k)\}_{k=0}^N, \quad \{D_k \in \mathbf{L}(\mathfrak{N}_k)\}_{k=0}^N, \quad N \leq \infty,$$

be the Schur parameters of M_0 and $\widetilde{M}_0$. The corresponding functions $M_k, \widetilde{M}_k \in \mathbf{N}_{\mathfrak{N}_k}^0$ satisfy the relations

$$(6.6) \quad \begin{aligned} M_k^{-1}(z) &= B_k - zI_{\mathfrak{N}_k} - D_k^{1/2}M_{k+1}(z)D_k^{1/2}, \\ \widetilde{M}_k^{-1}(z) &= B_k - zI_{\mathfrak{N}_k} - D_k^{1/2}\widetilde{M}_{k+1}(z)D_k^{1/2}, \quad k \geq 0, \end{aligned}$$

for all z from a common neighborhood of infinity, and have the asymptotic expansions

$$(6.7) \quad \begin{aligned} M_k(z) &= -I_{\mathfrak{N}_k}z^{-1} - B_kz^{-2} - (B_k^2 + D_k)z^{-3} + o(z^{-3}), \\ \widetilde{M}_k(z) &= -I_{\mathfrak{N}_k}z^{-1} - B_kz^{-2} - (B_k^2 + D_k)z^{-3} + o(z^{-3}), \quad z \rightarrow \infty. \end{aligned}$$

Our goal is to prove that $M_0 = \widetilde{M}_0$. If $N < \infty$, then $M_N^{-1}(z) = \widetilde{M}_N^{-1}(z) = B_N - zI_{\mathfrak{N}_N}$. Hence, relations (6.6) imply the equality $M_0 = \widetilde{M}_0$.

Let $N = \infty$. Let us mention that if H is a Hilbert space and V is an $\mathbf{L}(H)$ -valued function, holomorphic in a neighborhood of infinity and having the Taylor expansion

$$V(z) = -I_{\mathfrak{N}}z^{-1} - F_2z^{-2} - F_3z^{-3} - \cdots - F_nz^{-n} - \cdots,$$

then $V^{-1}(z) \in \mathbf{L}(H)$ and the function V^{-1} has the Taylor expansion

$$V^{-1}(z) = -zI_{\mathfrak{N}} + C_0 + C_1z^{-1} + C_2z^{-2} + C_3z^{-3} + \cdots + C_mz^{-m} + \cdots$$

in some neighborhood of infinity. The Taylor coefficients of V and V^{-1} are connected by the following relations:

$$(6.8) \quad \begin{cases} F_2 = C_0, \\ F_3 = C_0 F_2 + C_1 = C_0^2 + C_1, \\ F_4 = C_0 F_3 + C_1 F_2 + C_2, \\ F_5 = C_0 F_4 + C_1 F_3 + C_2 F_2 + C_3, \\ \dots\dots\dots \\ F_{n+1} = C_0 F_n + C_1 F_{n-1} + \dots + C_{n-2} F_2 + C_{n-1}, \\ \dots\dots\dots \end{cases}$$

Let

$$\begin{aligned} M_k(z) &= -I_{\mathfrak{N}_k} z^{-1} - \sum_{j \geq 2} F_j^{(k)} z^{-j}, \quad \widetilde{M}_k(z) = -I_{\mathfrak{N}_k} z^{-1} - \sum_{j \geq 2} \widetilde{F}_j^{(k)} z^{-j}, \\ M_k^{-1}(z) &= -z I_{\mathfrak{N}_k} + \sum_{j \geq 0} C_j^{(k)} z^{-j}, \quad \widetilde{M}_k^{-1}(z) = -z I_{\mathfrak{N}_k} + \sum_{j \geq 0} \widetilde{C}_j^{(k)} z^{-j}, \quad k \in \mathbb{N}_0. \end{aligned}$$

Then

$$\begin{aligned} F_2^{(k)} &= \widetilde{F}_2^{(k)} = B_k, \quad F_3^{(k)} = \widetilde{F}_3^{(k)} = B_k^2 + D_k, \\ C_0^{(k)} &= \widetilde{C}_0^{(k)} = B_k \end{aligned}$$

for all $k \in \mathbb{N}_0$.

Clearly, $M_0(z) - \widetilde{M}_0(z) = o(z^{-3})$, $z \rightarrow \infty$. Fix $k \geq 1$. Then (6.6) gives $C_1^{(k)} = \widetilde{C}_1^{(k)} = D_k$, $C_2^{(k)} = \widetilde{C}_2^{(k)} = D_k^{1/2} B_{k+1} D_k^{1/2}$, $C_3^{(k)} = \widetilde{C}_3^{(k)} = D_k^{1/2} (B_{k+1}^2 + D_{k+1}) D_k^{1/2}$. From (6.8) it follows that

$$F_4^{(k)} = \widetilde{F}_4^{(k)}, \quad F_5^{(k)} = \widetilde{F}_5^{(k)}.$$

Then we get $C_1^{(k-1)} = \widetilde{C}_1^{(k-1)}$, $C_2^{(k-1)} = \widetilde{C}_2^{(k-1)}$, $C_3^{(k-1)} = \widetilde{C}_3^{(k-1)}$, $C_4^{(k-1)} = \widetilde{C}_4^{(k-1)}$, $C_5^{(k-1)} = \widetilde{C}_5^{(k-1)}$, and relations (6.8) yield

$$F_4^{(k-1)} = \widetilde{F}_4^{(k-1)}, \quad \dots, \quad F_7^{(k-1)} = \widetilde{F}_7^{(k-1)}.$$

Arguing similarly we get the equalities

$$F_4^{(0)} = \widetilde{F}_4^{(0)}, \quad \dots, \quad F_{2k+3}^{(0)} = \widetilde{F}_{2k+3}^{(0)}.$$

Hence,

$$(6.9) \quad M_0(z) - \widetilde{M}_0(z) = o(z^{-2k-3}), \quad z \rightarrow \infty.$$

Since k is an arbitrary natural number, we conclude that $M_0 = \widetilde{M}_0$. ■

Remark 6.8. In the proof of Theorem 6.7 it is shown that if the Schur parameters of functions M_0 and $\tilde{M}_0$ from the class $\mathbf{M}_{\mathfrak{H}}^0$ are such that $B_l = \tilde{B}_l$ and $D_l = \tilde{D}_l$ for $l = 0, 1, \dots, k+1$, then (6.9) holds true. Similarly, if $B_l = \tilde{B}_l$ for $l = 0, 1, \dots, k+1$ and $D_l = \tilde{D}_l$ for $l = 0, 1, \dots, k$, then

$$(6.10) \quad M_0(z) - \tilde{M}_0(z) = o(z^{-2k-2}), \quad z \rightarrow \infty.$$

Using (6.8) in the form

$$\begin{cases} C_0 = F_2, \\ C_1 = F_3 - C_0 F_2, \\ C_2 = F_4 - C_0 F_3 - C_1 F_2, \\ \dots\dots\dots \\ C_{n-1} = F_{n+1} - C_0 F_n - C_1 F_{n-1} - \dots - C_{n-2} F_2, \\ \dots\dots\dots \end{cases}$$

one can prove the converse statements:

- (1) if (6.9) is satisfied, then $B_l = \tilde{B}_l$ and $D_l = \tilde{D}_l$ for $l = 0, 1, \dots, k+1$;
- (2) if (6.10) is satisfied, then $B_l = \tilde{B}_l$ for $l = 0, 1, \dots, k+1$ and $D_l = \tilde{D}_l$ for $l = 0, 1, \dots, k$.

7. Block operator Jacobi matrices

Let $\{H_k\}_{k=0}^N$ be a sequence of separable Hilbert spaces, $N \leq \infty$. Let $\{B_k \in \mathbf{L}(H_k)\}_{k=0}^N$ and $B_1, B_2, \dots$ be selfadjoint operators. Let $\{A_k \in \mathbf{L}(H_{k+1}, H_k)\}_{k=0}^N$ be a sequence of operators. Assume that

$$(7.1) \quad \sup\{\|B_k\| + \|A_k\|, k \in \mathbb{N}_0\} < \infty \quad \text{if } N = \infty.$$

Suppose also that

$$(7.2) \quad A_k^* A_k > 0 (\iff \ker A_k = \{0\}) \quad \text{for all } k.$$

Define a Hilbert space

$$\mathfrak{H} = \sum_{k=0}^N \oplus H_k, \quad N \leq \infty,$$

with inner product

$$(f, g)_{\mathfrak{H}} = \sum_{k=0}^N (f_k, g_k)_{H_k}$$

for $f = (f_0, f_1, \dots)^T$, $g = (g_0, g_1, \dots)^T$, $f_k, g_k \in H_k$, $k \geq 0$. Define in $\mathfrak{H}$ an operator J given by the finite (if $N < \infty$) or infinite (if $N = \infty$) three-diagonal matrix

$$(7.3) \quad J = \begin{pmatrix} B_0 & A_0 & 0 & 0 & 0 & \dots \\ A_0^* & B_1 & A_1 & 0 & 0 & \dots \\ 0 & A_1^* & B_2 & A_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We shall call a matrix of the form (7.3) a block operator Jacobi matrix.

Proposition 7.1. *The operator J is bounded, H_0 -quasi-selfadjoint, and H_0 -simple in the Hilbert space $\mathfrak{H}$. Moreover, let M_0 be the H_0 -Weyl function of J and let M_k be the H_k -Weyl function of the block Jacobi operator J_k acting in the Hilbert space $\mathfrak{H}_k = \sum_{j=k}^N \oplus H_j$ and given by the operator matrix obtained from J by crossing out the first k rows and the first k columns. Then the relations*

$$(7.4) \quad \begin{aligned} M_k(z) &= -I_{\mathfrak{H}_k} z^{-1} - B_k z^{-2} - (B_k^2 + A_k A_k^*) z^{-3} + o(z^{-3}), \quad z \rightarrow \infty, \\ B_k - z I_{\mathfrak{H}_k} - M_k^{-1}(z) &= A_k M_{k+1}(z) A_k^*, \quad |z| > \|J\|, \end{aligned}$$

hold for all k .

Proof. Obviously, condition (7.1) yields the boundedness of J .

Let $f = (f_0, 0, 0, \dots)^T$ and let

$$g^{(n)} = (g_0^{(n)}, g_1^{(n)}, \dots)^T := J^n f, \quad n \in \mathbb{N}_0.$$

Then

$$(7.5) \quad g_n^{(n)} = A_{n-1}^* A_{n-2}^* \cdots A_0^* f_0, \quad g_{n+1}^{(n)} = g_{n+2}^{(n)} = \dots = 0,$$

and condition (7.2) implies that the linear manifold $\text{span} \{J^n H_0, n \in \mathbb{N}_0\}$ is dense in $\mathfrak{H}$. Hence, J is an H_0 -simple operator. Because the operators $B_1, B_2, \dots$ are selfadjoint, the operator J is H_0 -quasi-selfadjoint.

It is sufficient to prove (7.4) for $k = 0$. To do this write the operator J in the matrix form

$$J = \begin{pmatrix} B_0 & \tilde{A}_0 \\ \tilde{A}_0^* & J_1 \end{pmatrix} : \begin{matrix} H_0 \\ \oplus \\ \mathfrak{H}_1 \end{matrix} \rightarrow \begin{matrix} H_0 \\ \oplus \\ \mathfrak{H}_1 \end{matrix},$$

where

$$\tilde{A}_0 = (A_0 \quad 0 \quad 0 \quad \dots) : \mathfrak{H}_1 \rightarrow H_0, \quad \tilde{A}_0^* = \begin{pmatrix} A_0^* \\ 0 \\ 0 \\ \vdots \end{pmatrix} : H_0 \rightarrow \mathfrak{H}_1.$$

This yields

$$M_0(z) = -I_{\mathfrak{H}_k} z^{-1} - B_0 z^{-2} - (B_0^2 + A_0 A_0^*) z^{-3} + o(z^{-3}), \quad z \rightarrow \infty,$$

and by Theorem 3.2 we have

$$\begin{aligned} B_0 - zI_{H_0} - M_0^{-1}(z) &= \tilde{A}_0(J_1 - zI_{\mathfrak{H}_1})^{-1} \tilde{A}_0^* = \\ &= A_0 P_{H_1} \left\{ (J_1 - zI_{\mathfrak{H}_1})^{-1} \upharpoonright H_1 \right\} A_0^* = A_0 M_1(z) A_0^*, \quad |z| > \|J\|. \end{aligned}$$

■

Let $T \in \mathbf{L}(\mathfrak{H})$ be $\mathfrak{N}$ -simple. For each $n \in \mathbb{N}_0$, define a subspace

$$(7.6) \quad \mathcal{L}_n = \overline{\text{span}} \{T^k \mathfrak{N}, \quad k = 0, 1, \dots, n\}.$$

Clearly,

$$\mathfrak{N} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots \subseteq \mathcal{L}_{n-1} \subseteq \mathcal{L}_n \subseteq \dots$$

and

$$(7.7) \quad T\mathcal{L}_n \subseteq \mathcal{L}_{n+1}, \quad n \in \mathbb{N}_0.$$

Let

$$(7.8) \quad \Omega_n = \mathcal{L}_n \ominus \mathcal{L}_{n-1}, \quad n \in \mathbb{N}.$$

Since T is $\mathfrak{N}$ -simple, we obtain

$$(7.9) \quad \mathfrak{H} = \mathfrak{N} \oplus \sum_{n \geq 1} \oplus \Omega_n.$$

Notice that $\mathfrak{H} = \mathfrak{N} \oplus \sum_{n \geq 1}^N \oplus \Omega_n$, where $N < \infty$, if and only if $\mathcal{L}_N = \mathfrak{H}$.

Proposition 7.2. *Let $T \in \mathbf{L}(\mathfrak{H})$ be an $\mathfrak{N}$ -simple and $\mathfrak{N}$ -quasi-selfadjoint operator. Then T has a block operator Jacobi representation (7.3) with respect to the decomposition (7.9).*

Proof. From (7.7) and (7.8) it follows that $T\Omega_n \perp \Omega_k$ for all $k \geq n + 2$. Because T is $\mathfrak{N}$ -quasi-selfadjoint, we get $\Omega_n \subset \text{Ker } T_I$ for all $n \in \mathbb{N}$. Hence, $T\Omega_n \perp \Omega_k$, $k = 0, 1, \dots, n - 2$, for $n \geq 2$. It follows that T admits a block operator Jacobi representation (7.3) with respect to the decomposition (7.9). ■

Let $\mathfrak{N}_0$ be a separable Hilbert space. Suppose that a finite or infinite chain of its nontrivial subspaces

$$\mathfrak{N}_0 \supseteq \mathfrak{N}_1 \supseteq \mathfrak{N}_2 \supseteq \dots$$

is given. Define a Hilbert space

$$\mathfrak{H} = \mathfrak{N}_0 \oplus \mathfrak{N}_1 \oplus \dots$$

Denote by j_k the embedding operator from $\mathfrak{N}_{k+1}$ into $\mathfrak{N}_k$, $k \geq 0$. Let $\{B_k \in \mathbf{L}(\mathfrak{N}_k)\}$ and suppose that $B_1, B_2, \dots$ are selfadjoint. Let $\{A_k \in \mathbf{L}(\mathfrak{N}_{k+1}, \mathfrak{N}_k)\}$ be a sequence of operators such that

- each operator $j_k A_k^* \in \mathbf{L}(\mathfrak{N}_k)$ is nonnegative selfadjoint in $\mathfrak{N}_k$,
- $\ker A_k^* = \mathfrak{N}_k \ominus \mathfrak{N}_{k+1}$ for all k .

Then we have $A_k^* A_k > 0$ for all $k \geq 0$. Suppose that condition (7.1) is satisfied.

The corresponding block operator matrix of the form (7.3) we shall call a *special block operator Jacobi matrix*.

Example 7.3. Let $\dim \mathfrak{N} = m < \infty$ and let $\mathfrak{N}_k = \mathfrak{N}$ for all k . The block Jacobi operator

$$J = \begin{pmatrix} 0 & \frac{1}{2}I_{\mathfrak{N}} & 0 & 0 & 0 & \dots \\ \frac{1}{2}I_{\mathfrak{N}} & 0 & \frac{1}{2}I_{\mathfrak{N}} & 0 & 0 & \dots \\ 0 & \frac{1}{2}I_{\mathfrak{N}} & 0 & \frac{1}{2}I_{\mathfrak{N}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called the discrete Schrödinger operator [10].

Proposition 7.4. *Every block operator Jacobi matrix is unitarily equivalent to a special block Jacobi operator matrix.*

Proof. Let J be of the form (7.3). Let us find the Schur parameters of the H_0 -Weyl function M_{J, H_0} of J in accordance with the construction of Section 6 and relations (7.4). Put

$$\widetilde{B}_0 = B_0, \quad \widetilde{D}_0 = A_0 A_0^*, \quad \mathfrak{N}_1 = \overline{\text{ran}} \widetilde{D}_0 = \overline{\text{ran}} A_0.$$

By the polar decomposition we have $A_0^* = V_1 \tilde{D}_0^{1/2}$, where V_1 is a unitary operator from $\mathfrak{N}_1$ onto H_1 . Then define

$$\tilde{B}_1 = V_1^* B_1 V_1, \quad \tilde{D}_1 = V_1^* A_1 A_1^* V_1, \quad \mathfrak{N}_2 = \overline{\text{ran}} \tilde{D}_1.$$

Clearly, $\mathfrak{N}_2 \subseteq \mathfrak{N}_1$ and there exists a unitary operator V_2 acting from $\mathfrak{N}_2$ onto H_2 such that $V_2 \tilde{D}_1^{1/2} = A_1^* V_1$. Continue this process. Then we get a chain of Hilbert spaces

$$\mathfrak{N}_0 = H_0 \supseteq \mathfrak{N}_1 \supseteq \cdots,$$

a sequence of unitary operators $V_k \in \mathbf{L}(\mathfrak{N}_k, H_k)$, $k \geq 1$, $V_0 := I_{\mathfrak{N}_0}$, and sequences of operators

$$\{\tilde{B}_k = V_k^* B_k V_k \in \mathbf{L}(\mathfrak{N}_k)\}_{k \geq 0}, \quad \{\tilde{D}_k = V_k^* V_{k-1}^* \cdots V_1^* V_0^* A_k A_k^* V_0 V_1 \cdots V_{k-1} V_k\}_{k \geq 0}.$$

These data are the Schur parameters of M_{J, H_0} . Now define the operators $\tilde{A}_k = \tilde{D}_k^{1/2} \upharpoonright \mathfrak{N}_{k+1} \in \mathbf{L}(\mathfrak{N}_{k+1}, \mathfrak{N}_k)$. Let $\tilde{J}_0$ be the special block operator Jacobi matrix constructed by means of $\{\tilde{B}_k\}$ and $\{\tilde{A}_k\}$. Then one can check that $\mathcal{V}^{-1} J \mathcal{V} = \tilde{J}_0$, where the unitary operator $\mathcal{V}$ from $\tilde{\mathfrak{H}} = \sum_{k \geq 0} \oplus \mathfrak{N}_k$ onto $\mathfrak{H} = \sum_{k \geq 0} \oplus H_k$ is given by the orthogonal sum of the operators V_k , i.e.,

$$\mathcal{V} = I_{\mathfrak{N}_0} \oplus \sum_{k \geq 1} \oplus V_k. \quad \blacksquare$$

Clearly, any function M from the class $\mathbf{M}_{\mathfrak{N}}^0$ corresponds to some special block operator Jacobi matrix

$$(7.10) \quad J_0 = \begin{pmatrix} B_0 & D_0^{1/2} & 0 & 0 & 0 & \cdots \\ D_0^{1/2} & B_1 & D_1^{1/2} & 0 & 0 & \cdots \\ 0 & D_1^{1/2} & B_2 & D_2^{1/2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \begin{matrix} \mathfrak{N}_0 \\ \oplus \\ \mathfrak{N}_1 \\ \oplus \\ \mathfrak{N}_2 \\ \oplus \\ \vdots \end{matrix} \rightarrow \begin{matrix} \mathfrak{N}_0 \\ \oplus \\ \mathfrak{N}_1 \\ \oplus \\ \mathfrak{N}_2 \\ \oplus \\ \vdots \end{matrix},$$

constructed by means of the Schur parameters of M . Observe that (see (7.5), (7.6), (7.8))

$$P_{\mathfrak{N}_n} J_0^n f = D_{n-1}^{1/2} D_{n-2}^{1/2} \cdots D_1^{1/2} D_0^{1/2} f, \quad f \in \mathfrak{N}_0,$$

$$\mathcal{L}_n^0 = \overline{\text{span}} \{J_0^k \mathfrak{N}_0, \quad k = 0, 1, \dots, n\} = \sum_{k=0}^n \oplus \mathfrak{N}_k,$$

$$\Omega_n^0 = \mathcal{L}_n^0 \ominus \mathcal{L}_{n-1}^0 = \mathfrak{N}_n.$$

Theorem 7.5. *Let T be an $\mathfrak{N}_0$ -quasi-selfadjoint and $\mathfrak{N}_0$ -simple operator in a separable Hilbert space $\mathfrak{H}$. Then T is $\mathfrak{N}_0$ -unitarily equivalent to an operator given by a special block Jacobi operator matrix (7.10), constructed by means of the Schur parameters of the $\mathfrak{N}$ -Weyl function $M_{T,\mathfrak{N}}$ of T .*

Proof. By Proposition 7.2, the operator T has a block Jacobi matrix representation with respect to the decomposition (7.9). Hence, according to Proposition 7.4 the operator T is $\mathfrak{N}$ -unitarily equivalent to the special block operator Jacobi matrix constructed by means of the Schur parameters of $M_{T,\mathfrak{N}}$. ■

Corollary 7.6. *If a bounded selfadjoint operator has a spectrum of multiplicity $m < \infty$ (see [2]), then it is unitarily equivalent to a special selfadjoint block operator Jacobi matrix.*

Corollary 7.7. *Each quasi-selfadjoint and completely non-selfadjoint operator T in a separable Hilbert space is $\overline{\text{ran}} T_I$ -unitarily equivalent to a special block operator Jacobi matrix.*

8. Livšic–Brodskiĭ characteristic operator-functions and Weyl functions

In this section we establish a connection between the class $\mathbf{M}_{\mathfrak{N}}^0$ and the class $\Omega_{\mathcal{J}}$ of characteristic functions of the Livšic–Brodskiĭ operator colligations.

Definition 8.1. ([20, §1]; see also [21], [37]) Let $\mathfrak{H}$ be a Hilbert space and let $T \in L(\mathfrak{H})$. The aggregate

$$(8.1) \quad \Theta = \begin{pmatrix} T & K & \mathcal{J} \\ \mathfrak{H} & & \mathfrak{E} \end{pmatrix}$$

is called the Livšic–Brodskiĭ operator colligation with the main operator T if $\mathfrak{E}$ is a Hilbert space, $\mathcal{J}$ is a unitary and selfadjoint operator in $\mathfrak{E}$, i.e., $\mathcal{J} = \mathcal{J}^*$ and $\mathcal{J}^2 = I_{\mathfrak{E}}$, $K \in L(\mathfrak{E}, \mathfrak{H})$, and the identity

$$(8.2) \quad T_I = K\mathcal{J}K^*.$$

holds.

The spaces $\mathfrak{H}$ and $\mathfrak{E}$ are called respectively *interior* and *exterior* spaces of the colligation Θ , and the operators T , K , $\mathcal{J}$ are called *main*, *channel*, and *directing* operators, respectively. Relation (8.2) yields the inclusion

$$\text{ran } T_I \subseteq \text{ran } K.$$

It is known (see [20, Theorem 1.1.]) that *if T is a bounded linear operator acting in a separable Hilbert space $\mathfrak{H}$, and $\mathfrak{N}$ is any subspace containing $\overline{\text{ran}} T_I$, then there exists a colligation Θ of the form (8.1) such that T is its main operator and $\overline{\text{ran}} K = \mathfrak{N}$.*

Let Θ be a colligation of the form (8.1) and let

$$\mathfrak{H}_\Theta = \overline{\text{span}} \{T^n K \mathfrak{E}, n \in \mathbb{N}_0\}, \quad \mathfrak{H}_\Theta^{(0)} = \mathfrak{H} \ominus \mathfrak{H}_\Theta.$$

Then the subspaces $\mathfrak{H}_\Theta$ and $\mathfrak{H}_\Theta^{(0)}$ reduce T and $T|_{\mathfrak{H}_\Theta^{(0)}}$ is a selfadjoint operator in $\mathfrak{H}_\Theta^{(0)}$ [20]. The colligation Θ is called *simple* if $\mathfrak{H}_\Theta = \mathfrak{H}$. Clearly, if T is completely non-selfadjoint, then the colligation is simple.

Given a colligation

$$\Theta = \begin{pmatrix} T & K & \mathcal{J} \\ \mathfrak{H} & & \mathfrak{E} \end{pmatrix},$$

the holomorphic $\mathbf{L}(\mathfrak{E})$ -valued function W_Θ , defined by

$$(8.3) \quad W_\Theta(z) = I - 2iK^*(T - zI)^{-1}K\mathcal{J}, \quad z \in \rho(T),$$

is called the *characteristic function* of the operator colligation Θ (see [20, §3]). It has the following properties:

$$(8.4) \quad \begin{aligned} W_\Theta(z)\mathcal{J}W_\Theta^*(\mu) - \mathcal{J} &= 2i(\bar{\mu} - z)K^*(T - zI_\mathfrak{H})^{-1}(T^* - \bar{\mu}I_\mathfrak{H})^{-1}K, \\ W_\Theta^*(\mu)\mathcal{J}W_\Theta(z) - \mathcal{J} &= 2i(\bar{\mu} - z)\mathcal{J}K^*(T^* - \bar{\mu}I_\mathfrak{H})^{-1}(T - zI_\mathfrak{H})^{-1}K\mathcal{J}, \\ &z, \mu \in \rho(T), \\ i(W_\Theta(z) + I_\mathfrak{H})^{-1}(W_\Theta(z) - I_\mathfrak{H})\mathcal{J} &= K^*(T_R - zI_\mathfrak{H})^{-1}K, \quad z \in \rho(T) \cap \rho(T_R). \end{aligned}$$

Two colligations

$$\Theta_1 = \begin{pmatrix} T_1 & K_1 & \mathcal{J} \\ \mathfrak{H}_1 & & \mathfrak{E} \end{pmatrix} \quad \text{and} \quad \Theta_2 = \begin{pmatrix} T_2 & K_2 & \mathcal{J} \\ \mathfrak{H}_2 & & \mathfrak{E} \end{pmatrix}$$

are called unitarily equivalent if there exists a unitary operator $U \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ such that

$$UT_1 = T_2U, \quad UK_1 = K_2.$$

Theorem 8.2. ([20, §3]) (1) *If Θ_1 and Θ_2 are unitarily equivalent, then their characteristic functions are equal.*

(2) *If the characteristic functions W_{Θ_1} and W_{Θ_2} of simple colligations Θ_1 and Θ_2 coincide in some neighborhood of infinity, then Θ_1 and Θ_2 are unitarily equivalent.*

Suppose that a linear operator $\mathcal{J}$, acting in a separable Hilbert space $\mathfrak{E}$, satisfies the conditions $\mathcal{J} = \mathcal{J}^*$ and $\mathcal{J}^2 = I_{\mathfrak{E}}$.

Definition 8.3. [20, page 43]. An $L(\mathfrak{E})$ -valued function W is said to belong to the class $\Omega_{\mathcal{J}}$ if it has the following properties:

- (1) W is holomorphic in some neighborhood G_W of infinity;
- (2) $\lim_{z \rightarrow \infty} \|W(z) - I_{\mathfrak{E}}\|_{\mathfrak{E}} = 0$;
- (3) for all $z \in G_W$ the operator $W(z) + I_{\mathfrak{E}}$ has a bounded inverse, while the function

$$(8.5) \quad V(z) = i(W(z) + I_{\mathfrak{E}})^{-1}(W(z) - I_{\mathfrak{E}})\mathcal{J} = i(W(z) - I_{\mathfrak{E}})(W(z) + I_{\mathfrak{E}})^{-1}\mathcal{J},$$

where $z \in G_W$, has a holomorphic continuation to an $L(\mathfrak{E})$ -valued Herglotz–Nevanlinna function from the class $N_{\mathfrak{E}}$.

Theorem 8.4. ([20, Theorem 5.1., Corollary 2, Corollary 3]) *Let an $L(\mathfrak{E})$ -valued function W be from the class $\Omega_{\mathcal{J}}$. Then there exists a simple colligation Θ with directing operator $\mathcal{J}$ such that its characteristic function W_{Θ} is equal to W in some neighborhood of infinity. The function W has an asymptotic expansion*

$$(8.6) \quad W(z) = I_{\mathfrak{E}} + 2iH\mathcal{J}z^{-1} + o(z^{-1}), \quad z \rightarrow \infty,$$

where $H = H^* \in L(\mathfrak{E})$ and $H \geq 0$. If $H = 0$, then $W \equiv I_{\mathfrak{E}}$.

If W_1 and W_2 belong to $\Omega_{\mathcal{J}}$, then $W_1W_2 \in \Omega_{\mathcal{J}}$.

Observe that from (8.3) it follows that

$$W_{\Theta}(z) = I_{\mathfrak{E}} - 2iK^*M_{T, \overline{\text{ran}} K}(z)K\mathcal{J}, \quad z \in \rho(T),$$

where $M_{T, \overline{\text{ran}} K}(z)$ is the $\overline{\text{ran}} K$ -Weyl function of the operator T . We shall prove the following statement.

Theorem 8.5. *Let $\mathfrak{E}$ be a Hilbert space and let $\mathcal{J} \in \mathbf{L}(\mathfrak{E})$, $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$. Assume that an $\mathbf{L}(\mathfrak{E})$ -valued function W belongs to the class $\Omega_{\mathcal{J}}$ and let*

$$(8.7) \quad H = \frac{1}{2i} \lim_{z \rightarrow \infty} z(W(z) - I_{\mathfrak{E}})\mathcal{J}.$$

Define $\mathfrak{N}_0 = \overline{\text{ran}} H$. Then there exists a function $M \in \mathbf{M}_{\mathfrak{N}_0}^0$ such that for any z in a neighborhood of infinity one has

$$W(z) = I_{\mathfrak{E}} - 2iH^{1/2}M(z)H^{1/2}\mathcal{J}.$$

Moreover, there exists a simple operator colligation Θ whose exterior space $\mathfrak{E}$ and directing operator are equal to $\mathfrak{E}$ and $\mathcal{J}$, resp., whose main operator J_0 is $\mathfrak{N}_0$ -simple, $\mathfrak{N}_0$ -quasi-selfadjoint and given by a special block operator Jacobi matrix, and whose characteristic function is equal to W .

Proof. Let V be defined by (8.5). Then

$$V(z)(W(z) + I_{\mathfrak{E}}) = i(W(z) - I_{\mathfrak{E}})\mathcal{J}$$

for z in some neighborhood of infinity. By Definition 8.3, the function V belongs to the class $\mathbf{N}_{\mathfrak{E}}$. Taking into account (8.6) we get

$$V(\infty) = 0, \quad \lim_{z \rightarrow \infty} zV(z) = -H.$$

By Theorem 5.6, the function V can be written in the form

$$V = H^{1/2}NH^{1/2},$$

where $N \in \mathbf{N}_{\mathfrak{N}_0}^0$. By Theorems 5.3 and 7.5, the function N is the $\mathfrak{N}_0$ -Weyl function of an $\mathfrak{N}_0$ -simple selfadjoint operator T_0 in a Hilbert space $\mathfrak{H}$ given by a special block operator Jacobi matrix constructed by means of the Schur parameters of N . Define the operator J_0 in $\mathfrak{H}$ by the equality

$$J_0 := T_0 + iH^{1/2}\mathcal{J}H^{1/2}P_{\mathfrak{N}_0},$$

where $P_{\mathfrak{N}_0}$ is the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{N}_0$. Clearly, the operator J_0 is $\mathfrak{N}_0$ -quasi-selfadjoint and takes the form of a special Jacobi matrix. Furthermore, define $K \in \mathbf{L}(\mathfrak{E}, \mathfrak{H})$ by the equality

$$K := H^{1/2}.$$

Then $K^* = H^{1/2}P_{\mathfrak{N}_0} \in \mathbf{L}(\mathfrak{H}, \mathfrak{E})$ and $(J_0)_I = K\mathcal{J}K^*$. Define the operator colligation

$$\Theta = \begin{pmatrix} J_0 & K & \mathcal{J} \\ \mathfrak{H} & & \mathfrak{E} \end{pmatrix}.$$

According to Proposition 4.6, the operator J_0 is $\mathfrak{N}_0$ -simple. Hence, the colligation Θ is simple. Its characteristic function W_Θ is given by

$$W_\Theta(z) = I_{\mathfrak{E}} - 2iH^{1/2}P_{\mathfrak{N}_0}R_{J_0}(z)H^{1/2}\mathcal{J}, \quad z \in \rho(J_0).$$

Let M be the $\mathfrak{N}_0$ -Weyl function of J_0 . Then M belongs to the class $\mathbf{M}_{\mathfrak{N}_0}^0$ and we have

$$W_\Theta(z) = I_{\mathfrak{E}} - 2iH^{1/2}M(z)H^{1/2}\mathcal{J}.$$

The last property from (8.4) gives

$$V_\Theta(z) = K^*R_{T_0}(z)K = H^{1/2}P_{\mathfrak{N}_0}R_{T_0}(z)H^{1/2} = H^{1/2}N(z)H^{1/2} = V(z)$$

for z in a neighborhood of infinity. On the other hand, from (8.5) it follows

$$\begin{aligned} W(z) &= (I_{\mathfrak{E}} + iV(z)\mathcal{J})^{-1}(I_{\mathfrak{E}} - iV(z)\mathcal{J}) = (I_{\mathfrak{E}} + iV_\Theta(z)\mathcal{J})^{-1}(I_{\mathfrak{E}} - iV_\Theta(z)\mathcal{J}) \\ &= W_\Theta(z). \end{aligned}$$

This completes the proof. ■

9. Quasi-selfadjoint discrete time-invariant systems

Let $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{H}$ be separable Hilbert spaces and let

$$(9.1) \quad T = \begin{pmatrix} B_0 & F \\ C & A \end{pmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathcal{H} \end{array} \rightarrow \begin{array}{c} \mathfrak{N} \\ \oplus \\ \mathcal{H} \end{array}$$

be a bounded linear operator. The system of equations

$$(9.2) \quad \begin{cases} h_{k+1} = Ah_k + C\xi_k, \\ \sigma_k = Fh_k + B_0\xi_k, \end{cases} \quad k \geq 0,$$

describes the evolution of a *linear discrete time-invariant system* $\tau = \{T, \mathfrak{M}, \mathfrak{N}, \mathcal{H}\}$. The Hilbert spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the input and the output spaces, respectively, and the Hilbert space $\mathcal{H}$ is called the state space. The operators A , C , F ,

and B_0 are called the *main operator*, the *control operator*, the *observation operator*, and the *feedthrough operator* of τ , respectively. The subspaces

$$(9.3) \quad \mathcal{H}^c = \overline{\text{span}} \{ A^n C \mathfrak{M}, n \in \mathbb{N}_0 \} \text{ and } \mathcal{H}^o = \overline{\text{span}} \{ A^{*n} F^* \mathfrak{N}, n \in \mathbb{N}_0 \}$$

are called the *controllable* and *observable* subspaces of $\tau = \{T, \mathfrak{M}, \mathfrak{N}, \mathcal{H}\}$, respectively. If $\mathcal{H}^c = \mathcal{H}$ ($\mathcal{H}^o = \mathcal{H}$), then the system τ is said to be *controllable* (*observable*). It is called *minimal* if it is both controllable and observable. If $\mathcal{H} = \text{clos} \{ \mathcal{H}^c + \mathcal{H}^o \}$, then the system τ is said to be *simple*. Two discrete-time systems $\tau_1 = \{T_1, \mathfrak{M}, \mathfrak{N}, \mathcal{H}_1\}$ and $\tau_2 = \{T_2, \mathfrak{M}, \mathfrak{N}, \mathcal{H}_2\}$ are *unitarily similar* if there exists a unitary operator U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that

$$(9.4) \quad A_2 = U A_1 U^*, \quad C_2 = U C_1, \quad F_2 = F_1 U^*, \text{ and } B_2 = B_1.$$

The function

$$(9.5) \quad \Theta_\tau(z) := B_0 - F R_A(z) C, \quad z \in \rho(A)$$

is called the *transfer function* of the system τ . Clearly, $\Theta(\infty) = B$. D.Z. Arov [14] has shown that two minimal passive systems τ_1 and τ_2 with the same transfer function Θ are only *weakly similar*, i.e., there is a closed densely defined operator $Z : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that Z is invertible, Z^{-1} is densely defined, and

$$(9.6) \quad Z A_1 f = A_2 Z f, \quad F_1 f = F_2 Z f, \quad f \in \text{dom } Z, \quad \text{and} \quad Z C_1 = C_2.$$

Definition 9.1. (Cf. [12]) The system $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathcal{H}\}$ is said to be *quasi-selfadjoint* if the operator T in (9.1) is $\mathfrak{N}$ -quasi-selfadjoint.

It follows that A is a selfadjoint operator in $\mathfrak{H}$ and $F = C^*$. Hence the controllable and observable subspaces of a quasi-selfadjoint system coincide. Proposition 4.1 implies that a quasi-selfadjoint system τ is minimal if and only if the operator T is $\mathfrak{N}$ -simple. Let Θ be the transfer function of a quasi-selfadjoint system τ . Since A is selfadjoint, the function

$$N(z) = B_0 - \Theta(z) = C^* R_A(z) C, \quad z \in \rho(A),$$

belongs to the class $\mathbf{N}_{\mathfrak{N}}$ and $N(\infty) = 0$. Set

$$D_0 = - \lim_{z \rightarrow \infty} z N(z) = \lim_{z \rightarrow \infty} z (\Theta(z) - \Theta(\infty))$$

and $\mathfrak{N}_1 = \overline{\text{ran}} D_0$. According to Theorem 5.6, there exists a function $N_1 \in \mathcal{N}_{\mathfrak{N}_1}^0$ such that $N = D_0^{1/2} N_1 D_0^{1/2}$. Observe that by Theorem 3.2 the function

$$M(z) = (\Theta(z) - zI_{\mathfrak{N}})^{-1}, \quad z \in \rho(T) \cap \rho(A),$$

is the $\mathfrak{N}$ -Weyl function of the $\mathfrak{N}$ -quasi-selfadjoint operator T . It follows that two minimal quasi-selfadjoint systems, whose transfer functions coincide in a neighborhood of infinity, are unitarily equivalent (see [12]). Notice that such kind of systems, which are called normal symmetric dynamical systems, have been studied in [19]. Passive quasi-selfadjoint discrete time-invariant systems ($\|T\| \leq 1$) have been investigated in [12].

Let

$$\{\mathfrak{N}_k\}_{k \geq 1}, \{B_k\}_{k \geq 1}, \{D_k\}_{k \geq 1}$$

be the Schur parameters of N_1 . Set $\mathcal{H}_0 = \sum_{j \geq 1} \oplus \mathfrak{N}_j$, $A_k = D_k^{1/2} \upharpoonright \mathfrak{N}_{k+1}$, $k \geq 1$, and let J_0 be a special block Jacobi matrix of the form (7.10) acting in the Hilbert space $\mathfrak{H}_0 = \mathfrak{N} \oplus \mathcal{H}_0$. From Theorem 7.5 we get the following statement.

Theorem 9.2. *Let $\tau = \{T; \mathfrak{N}, \mathfrak{N}, \mathcal{H}\}$ be a minimal quasi-selfadjoint system. Then τ is unitarily equivalent to the quasi-selfadjoint system of the form $\tau_0 = \{J_0; \mathfrak{N}, \mathfrak{N}, \mathcal{H}_0\}$.*

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Y. ARLINSKIĬ, Department of Mathematical Analysis, East Ukrainian National University, Kvartal Molodyozhny 20-A, Lugansk 91034, Ukraine; *e-mail*: yury_arlinskii@yahoo.com; yma@snu.edu.ua

L. KLOTZ, Institut für Mathematik, Leipzig University, PF 100920, Leipzig 04109, Germany; *e-mail*: klotz@mathematik.uni-leipzig.de